Automorphisms of irreducible symplectic manifolds

Von der Fakultät für Mathematik und Physik
der Gottfried Wilhelm Leibniz Universität Hannover
zur Erlangung des Grades

Doktor der Naturwissenschaften
Dr. rer. nat.

genehmigte Dissertation
von

Dipl.-Math. Malek Joumaah
geboren am 09.08.1986 in Hameln

2015

Referent:	Prof. Dr. Klaus Hulek, Hannover
Korreferentin:	Prof. Dr. Alessandra Sarti, Poitiers
Tag der Promotion:	16. Dezember 2014

Kurzzusammenfassung

Wir untersuchen Automorphismen von irreduziblen holomorph-symplektischen Mannigfaltigkeiten, einer Verallgemeinerung von K3-Flächen in höherer Dimension. Automorphismen von K3-Flächen bilden ein viel untersuchtes Thema, und es ist eine naheliegende Problemstellung, die Ergebnisse auf irreduzible symplektische Mannigfaltigkeiten zu verallgemeinern. Im ersten Teil dieser Arbeit geht es um Automorphismen der Ordnung 3 von 4-dimensionalen irreduziblen symplektischen Mannigfaltigkeiten. Wir betrachten Beispiele und wenden die holomorphe Lefschetz-Formel an, um topologische Informationen über den Fixort zu erhalten. Der größte Teil dieser Arbeit beschäftigt sich mit Modulräumen von Paaren (X, i), wobei X eine Deformation des Hilbert-Schemas von n Punkten auf einer K3-Fläche ist, und $i : X \to X$ eine nicht-symplektische Involution. Wir geben eine gittertheoretische Beschreibung der Deformationstypen solcher Paare an. Des Weiteren zeigen wir, dass ein quasi-projektiver Modulraum für eine gewisse Klasse solcher Involutionen existiert.

Schlagworte: irreduzible symplektische Mannigfaltigkeiten, Automorphismen, nicht-symplektische Involutionen

Abstract

We study automorphisms of irreducible holomorphic symplectic manifolds, which are higher dimensional generalizations of K3 surfaces. Automorphisms of K3 surfaces is a widely studied subject and it is a natural problem to generalize the results to irreducible symplectic manifolds. The first part of this thesis is concerned with automorphisms of order 3 on irreducible symplectic fourfolds. We use the holomorphic Lefschetz formula to obtain topological information about the fixed locus and consider some examples. The main part deals with moduli spaces of pairs (X, i), where X is an irreducible symplectic manifold deformation equivalent to the Hilbert scheme of n points on a K3 surface, and $i : X \to X$ is a non-symplectic involution. We give a lattice theoretic description of the deformation types of such pairs. Moreover, we show that there exists a quasi-projective moduli space for a certain class of involutions.

Keywords: irreducible symplectic manifolds, non-symplectic involutions, automorphisms

Contents

Introduction

This thesis is concerned with automorphisms of irreducible symplectic manifolds (also called hyperkähler manifolds), which form one of three types of manifolds occuring in the Beauville–Bogomolov decomposition of compact Kähler manifolds with trivial real first Chern class (or equivalently, compact Ricci-flat Kähler manifolds). The complex dimension of irreducible symplectic manifolds is always even, and in dimension 2 they coincide with K3 surfaces. Therefore, irreducible symplectic manifolds can be considered as higher dimensional generalizations of K3 surfaces.

First examples were given by Beauville [Bea83b], who showed that for every integer $n \geq 2$, the Hilbert scheme $S^{[n]}$ of n points on a K3 surface S is an irreducible symplectic manifold of dimension $2n$. Complex manifolds which are deformation equivalent to $S^{[n]}$ are also irreducible symplectic and are called of $K3^{[n]}$-type. Beauville gave another series of examples, the generalized Kummer varieties $K_n(A)$ of a complex 2-dimensional torus A. Up to deformation, the only other known examples of irreducible symplectic manifolds were constructed by O'Grady. These are desingularized moduli spaces of sheaves on K3 surfaces and abelian surfaces, and are of dimension 10 and 6 respectively.

Automorphisms of K3 surfaces is a widely studied topic, and it is a natural problem to generalize the results to irreducible symplectic manifolds. An important tool for the study of K3 surfaces is the Global Torelli theorem, which states that a K3 surface can be recovered from the Hodge structure of the group $H^2(S, \mathbb{Z})$ together with its lattice structure, which is defined by the intersection product. Moreover, using the strong form of the Global Torelli theorem, under certain conditions isometries of the lattice can be lifted to automorphisms of the surface. Together with the surjectivity of the period map, this reduces the theory of automorphisms to a certain extent to lattice theory. This was used extensively by Nikulin [Nik80a][Nik80b][Nik83] and others to study finite automorphism groups of K3 surfaces.

For an irreducible symplectic manifold X, the group $H^2(X, \mathbb{Z})$ carries a natural lattice structure defined by the Beauville–Bogomolov form, which generalizes the intersection form of $K3$ surfaces. The Local Torelli theorem was proved by Beauville [Bea83b] and the surjectivity of the period map by Huybrechts [Huy99].

The obvious generalization of the Global Torelli theorem turned out to be false, however, as shown by counterexamples given by Debarre [Deb84] and Namikawa [Nam02]. A correct formulation of the Global Torelli theorem for irreducible symplectic manifolds has been proved only recently by Verbitsky [Ver13].
In Chapter 1, we will give an overview of these results, in particular of the Global Torelli theorem and several of its implications, which have been shown by Markman.
Due to the importance of the Beauville–Bogomolov form, we will need some results from lattice theory. An overview will be given in Chapter 2.
A number of results about automorphisms of irreducible symplectic manifolds have been obtained over the last years. Boissière–Nieper-Wißkirchen–Sarti [BNWS11] and Oguiso–Schröer [OS11] gave examples of generalized Kummer varieties of dimension 4 and 6 with fixed-point free automorphisms of order 3 and 4, respectively. Their quotients can be considered as higher dimensional generalizations of Enriques surfaces. The fixed locus of involutions of $K3^{[2]}$-type manifolds has been systematically analyzed by Beauville [Bea11] in the non-symplectic case, and by Camere [Cam12] and Mongardi [Mon12] in the symplectic case. In Chapter 3, we consider some examples of automorphisms of order 3, and we apply the holomorphic Lefschetz formula to the non-symplectic case. We obtain the following result:

Proposition (Proposition 3.2.6). *Let X be a $K3^{[2]}$-type manifold and $f : X \to X$ a non-symplectic automorphism of order 3. The fixed locus X^f consists of N isolated points, the disjoint union C of smooth curves, and the disjoint union S of smooth surfaces. Moreover,* $\operatorname{tr} f^*|H^{1,1}(X) = 3s$ *for some integer $-3 \leq s \leq 7$, and*

$$\begin{aligned} 2N + \chi(C) &= 3s(s+3) \\ \chi(C) + 2c_2(S) &= 6s^2 \\ c_1^2(S) + c_2(S) &= 6s(s-1). \end{aligned}$$

Furthermore, any $-3 \leq s \leq 7$ occurs for some automorphism.

The main part of this thesis is Chapter 4, which is concerned with moduli spaces of manifolds X of $K3^{[n]}$-type with non-symplectic involutions $i : X \to X$. The most important deformation invariant of (X, i) is the invariant sublattice

$$H^2(X,\mathbb{Z})^i = \{h \in H^2(X,\mathbb{Z}) : i^*(h) = h\} \subset H^2(X,\mathbb{Z}).$$

More precisely, if Y is another manifold of $K3^{[n]}$-type and $j : Y \to Y$ is a non-symplectic involution such that (X, i) and (Y, j) are deformation equivalent, then there exists a parallel transport operator $g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$ such that $g(H^2(X,\mathbb{Z})^i) = H^2(Y,\mathbb{Z})^j$. If conversely such a parallel transport operator exists, then we will call (X, i) and (Y, j) of the same *lattice type*.

For K3 surfaces, Nikulin showed that the isometry class of the invariant sublattice determines the deformation type of an involution. For $K3^{[n]}$-type manifolds, even being of the same lattice type does not imply deformation equivalence. In order to obtain a criterion for deformation equivalence, we introduce the *stable invariant Kähler cone* $\widetilde{\mathcal{K}}_X^i \subset H^{1,1}(X,\mathbb{R})^i$ of (X,i). This is a cone containing the invariant Kähler cone $\mathcal{K}_X^i$ and consists of classes which deform into an invariant Kähler class for a generic small deformation of (X,i).

Theorem (Proposition 4.8.3 and Theorem 4.8.10)**.** *Let X and Y be manifolds of $K3^{[n]}$-type, and $i : X \to X$ and $j : Y \to Y$ be non-symplectic involutions. The pairs (X,i) and (Y,j) are deformation equivalent if and only if there exists a parallel transport operator*

$$g : H^2(X,\mathbb{Z}) \to H^2(Y,\mathbb{Z})$$

mapping $H^2(X,\mathbb{Z})^i$ to $H^2(Y,\mathbb{Z})^j$ and $\widetilde{\mathcal{K}}_X^i$ to $\widetilde{\mathcal{K}}_Y^j$

Now let L_n be the $K3^{[n]}$ lattice and fix a sublattice $M \subset L_n$ which is isometric isometric to $H^2(X,\mathbb{Z})^i \subset H^2(X,\mathbb{Z})$. We call involutions of the same lattice type as (X,i) *of type M*.

In order to obtain a purely lattice theoretic description of the deformation types of pairs of type M, we define lattice theoretic counterparts of the stable invariant Kähler cones, the *Kähler-type chambers* of M. There exists a group Γ_M acting on the set $\mathrm{KT}(M)$ of Kähler-type chambers of M such that the stable invariant Kähler cone of (X,i) defines an equivalence class in $\mathrm{KT}(M)/\Gamma_M$. Using the Global Torelli theorem and the preceding Theorem, we obtain the following result.

Theorem (Theorem 4.8.11)**.** *There exists a bijection between deformation types of involutions of type M and* $\mathrm{KT}(M)/\Gamma_M$.

For K3 surfaces, there exists a quasi-projective coarse moduli space of pairs of type M, which is a Zariski-open subset of an arithmetic quotient $\Omega^+_{M^\perp}/\Gamma^+_{M^\perp}$ of a bounded symmetric domain $\Omega^+_{M^\perp}$. We will see that in the $K3^{[n]}$case, a Hausdorff moduli space does not always exist. In order to obtain a quasi-projective, and in particular Hausdorff, moduli space, we will therefore restrict to the following class of involutions.

Definition 1. Let $i : X \to X$ be a non-symplectic involution. The pair (X,i) is called *simple*, if $\widetilde{\mathcal{K}}_X^i = \mathcal{K}_X^i$.

We show that simple pairs form the complement of a codimension 1 subvariety of the local deformation space and obtain the following result.

Theorem (Theorem 4.9.5)**.** *There exists a Zariski-open subset of an arithmetic quotient $\Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}}$, which is a coarse moduli space of simple pairs of type M and deformation type $[\mathcal{K}] \in \mathrm{KT}(M)/\Gamma_M$.*

A classification of invariant sublattices of non-symplectic involutions in the $K3^{[2]}$ case has been given in [BCS14]. In Chapter 5, we consider the $K3^{[n]}$ case for $n > 2$. In Theorem 5.0.1 we determine the discriminant group of invariant sublattices and give a partial classification of their isometry classes.

Notations and Definitions

Lattices. We will use some concepts about lattices in Chapter 1, before giving a more detailed overview in Chapter 2.

A *lattice* is a finitely generated abelian group L together with a non-degenerate symmetric bilinear form $(\cdot,\cdot) : L \times L \to \mathbb{Z}$. The rank of L is denoted by $r(L)$.

An *isometry* $L \to L'$ between two lattices is a group isomorphism preserving the bilinear forms. The group of isometries $L \to L$ is denoted by $O(L)$. For any field $\mathbb{K}$ we consider the $\mathbb{K}$-vector space $L_{\mathbb{K}} := L \otimes \mathbb{K}$ together with the induced $\mathbb{K}$-valued bilinear form. For an isometry $\sigma \in O(L)$, we also denote by $\sigma : L_{\mathbb{K}} \to L_{\mathbb{K}}$ the map obtained by linear extension.

The bilinear form defines an embedding $L \hookrightarrow L^* := \mathrm{Hom}(L, \mathbb{Z})$. The lattice L is called unimodular, if $L = L^*$. The hyperbolic plane U is the unimodular lattice of signature $(1, 1)$. We denote by E_8 the unimodular negative definite lattice of rank 8. The rank 1 lattice generated by an element v with $(v, v) = k$ is denoted by $\langle k\rangle$.

We denote by $L \oplus M$ the orthogonal direct sum of two lattices. The orthogonal complement of a sublattice $M \subset L$ is given by

$$M^{\perp} := \{v \in L : (v, w) = 0 \text{ for every } w \in M\}.$$

Let X be a complex manifold.

- TX is the holomorphic tangent bundle of X.
- $N_{Y/X}$ is the normal bundle of a complex submanifold $Y \subset X$.
- $\Omega^k_X = \Lambda^k(TX)^*$ is the sheaf of holomorphic k-forms.
- $H^{p,q}(X) = H^q(\Omega^p_X)$.
- $h^{p,q}(X) = \dim_{\mathbb{C}} H^{p,q}(X)$.
- $c_k(V) \in H^{2k}(X, \mathbb{Z})$ is the k-th Chern class of a vector bundle V on X.
- $c_k(X) = c_k(TX)$.

Ackowledgements

I am grateful to my advisor Klaus Hulek for suggesting this topic and for many helpful discussions. Moreover, I would like to thank M. Wandel and C. Camere for interesting discussions and useful remarks, and D. Ploog for reading a part of this work. I am also grateful to A. Sarti for inviting me to Poitiers.

Chapter 1

Irreducible symplectic manifolds

In this chapter we introduce irreducible symplectic manifolds und present some of the results that we will need later, in particular the Global Torelli theorem and its consequences.

1.1 Definiton and Examples

Definition 1.1.1. An *irreducible (holomorphic) symplectic manifold* is a complex manifold X, such that

(i) X is a compact Kähler manifold,

(ii) X is simply connected,

(iii) $H^0(X, \Omega_X^2) = \mathbb{C}\omega$, where ω is an everywhere non-degenerate holomorphic 2-form on X.

The form ω is also called the *symplectic* form of X. The non-degeneracy of ω implies that X has even complex dimension $2n$. Moreover, by [Bea83b, Prop. 3] one has

$$H^0(X, \Omega_X^k) = \begin{cases} \mathbb{C} \cdot \omega^{k/2}, & \text{if } k \text{ is even,} \\ 0, & \text{if } k \text{ is odd.} \end{cases}$$

Since ω^n is nowhere vanishing, this implies in particular that the canonical bundle $K_X \cong \mathcal{O}_X$ is trivial, and hence that $c_1(X) = 0$. In fact, irreducible symplectic manifolds are one of three basic types of compact Kähler manifolds with vanishing first Chern class:

Theorem 1.1.2 (Beauville–Bogomolov decomposition)**.** *Let X be a compact Kähler manifold such that $c_1(X)_{\mathbb{R}} = 0$. Then there exists a finite étale covering of X which is a product of tori, Calabi–Yau manifolds, and irreducible symplectic manifolds.*

Example 1.1.3. (i) A surface is an irreducible symplectic manifold if and only if it is a K3 surface. Therefore, irreducible symplectic manifolds can be considered as higher dimensional generalizations of K3 surfaces.

(ii) Let S be a K3 surface and $S^{[n]}$ the Hilbert scheme (or Douady space, if S is not projective) of length n subschemes of S. Then $S^{[n]}$ is an irreducible symplectic manifold of dimension $2n$. For $n = 2$, this was first shown by Fujiki. In this case, $S^{[2]} \to S^{(2)}$ is simply the blow-up of the symmetric square $S^{(2)} = (S \times S)/\mathfrak{S}_2$ along the diagonal. Beauville showed in [Bea83b] that $S^{[n]}$ is irreducible symplectic for arbitrary n, thereby giving an example of an irreducible symplectic manifold in every possible dimension.

(iii) Let A be a 2-dimensional complex torus. The Hilbert scheme of points on A admits a symplectic form, but as A itself, it is not simply-connected. However, consider the summation map

$$\begin{array}{cccc} s: & A^{[n+1]} & \longrightarrow & A \\ & Z & \longmapsto & \sum_{p\in A} l(\mathcal{O}_{Z,p})p. \end{array}$$

As shown by Beauville, the fibre $K_n(A) := s^{-1}(0)$ is an irreducible symplectic manifold of dimension $2n$. Therefore, this gives a second series of examples that exist in every even dimension. Since $K_1(A)$ is the Kummer K3 surface of A, the manifolds $K_n(A)$ are called *generalized Kummer varieties.*

(iv) O'Grady constructed examples in dimensions 6 and 10 as desingularized moduli spaces of sheaves on abelian surfaces [O'G03] and K3 surfaces [O'G99], respectively.

Further examples of irreducible symplectic manifolds can be obtained by deformation:

Theorem 1.1.4. *Let $\pi : \mathcal{X} \to S$ be a smooth and proper family over a connected analytic space S. If $X_0 = \pi^{-1}(0)$ is an irreducible symplectic manifold, then for any $t \in S$ the fibre $X_t = \pi^{-1}(t)$ is an irreducible symplectic manifold if it is Kähler.*

Proof. [Bea83b, Prop. 9 and Rem. 10] □

In fact, all known irreducible symplectic manifolds are deformations of one of the manifolds given in Example 1.1.3.

Definition 1.1.5. A manifold is called of *$K3^{[n]}$-type*, if it is deformation equivalent to $S^{[n]}$ for some (and hence for any) K3 surface S.

1.2 The Beauville–Bogomolov form

For a K3 surface S, the Hodge structure and the intersection form on $H^2(S,\mathbb{Z})$ contain most information about S. The Beauville–Bogomolov form, which we describe in this section, is a natural way to generalize the intersection form to higher dimensional irreducible symplectic manifolds.

Let $\omega \in H^0(X,\Omega_X^2)$ be a symplectic form with $\int_X (\omega\overline{\omega})^n = 1$, and for $\alpha \in H^2(X,\mathbb{C})$ let

$$q'_X(\alpha) = \frac{n}{2}\int_X (\omega\overline{\omega})^{n-1}\alpha^2 + (1-n)\int_X \omega^{n-1}\,\overline{\omega}^n\alpha \cdot \int_X \omega^n\,\overline{\omega}^{n-1}\alpha.$$

Theorem 1.2.1 (Beauville). *There exists a positive real number c_X such that $q_X := c_X \cdot q'_X$ is a non-degenerate primitive integral quadratic form on $H^2(X,\mathbb{Z})$ of signature $(3, b_2(X)-3)$. Furthermore, one has*

$$q_X(\omega) = 0, \quad q_X(\omega+\overline{\omega}) > 0.$$

Proof. [Bea83b, Thm. 5] □

The quadratic form q_X is called the *Beauville–Bogomolov* form (or sometimes *Beauville–Bogomolov–Fujiki* form). We denote the corresponding symmetric bilinear form by $(\cdot,\cdot)_X$, or simply $(\cdot,\cdot)$.

Theorem 1.2.2 (Fujiki). *There exists a positive rational number c'_X such that*

$$q_X(\alpha)^n = c'_X \int_X \alpha^{2n}$$

for every $\alpha \in H^2(X,\mathbb{Z})$. Furthermore, the number c'_X only depends on the deformation type of X.

Proof. [Fuj87] □

The following properties of the Beauville–Bogomolov form are immediate consequences of its definition or of Fujiki's theorem:

(i) For any small deformation $\pi : \mathcal{X} \to S$ of $X = \pi^{-1}(0)$ and any $t \in S$, the isomorphism $H^2(X,\mathbb{Z}) \cong H^2(X_t,\mathbb{Z})$ obtained by parallel transport in the local system $R^2\pi_*\mathbb{Z}$ preserves the Beauville–Bogomolov form. In particular, the isometry class of $H^2(X,\mathbb{Z})$ only depends on the deformation type of X.

(ii) Let

$$H^2(X,\mathbb{C}) = H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$$

be the Hodge decomposition of $H^2(X,\mathbb{C})$. With respect to the Beauville–Bogomolov form, the space $(H^{2,0}(X) \oplus H^{0,2}(X))$ is orthogonal to $H^{1,1}(X)$. Using

$$q_X(\omega + \overline{\omega}) > 0,$$

this implies

$$H^{1,1}(X) = (H^{2,0}(X) \oplus H^{0,2}(X))^{\perp} \subset H^2(X,\mathbb{C}).$$

In particular, since $H^2(X,\mathbb{Z})$ is invariant under complex conjugation, we have

$$\mathrm{NS}(X) = H^{1,1}(X,\mathbb{Z}) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X) = H^2(X,\mathbb{Z}) \cap \omega^{\perp}.$$

(iii) Any Kähler class x on X satisfies $q_X(x) > 0$.

As for K3 surfaces, there is a numerical criterion for projectivity.

Theorem 1.2.3 (Huybrechts)**.** *An irreducible symplectic manifold X is projective if and only if there exists a line bundle $\mathcal{L}$ on X with $q_X(c_1(\mathcal{L})) > 0$.*

Proof. [Huy99, Thm. 3.11] □

Example 1.2.4. If S is a K3 surface, then the Beauville–Bogomolov coincides with the intersection form, that is,

$$H^2(S,\mathbb{Z}) \cong L_{K3} := 3U \oplus 2E_8.$$

Example 1.2.5. Let $S^{(n)} := S^n/\mathfrak{S}_n$ be the n-th symmetric product of S. The singular locus of $S^{(n)}$ is the large diagonal $\Delta \subset S^{(n)}$. Writing elements of $S^{(n)}$ as formal sums, the *Hilbert–Chow morphism*

$$\begin{aligned} \varepsilon : S^{[n]} &\to S^{(n)} \\ Z &\mapsto \sum_{p \in S} l(\mathcal{O}_{Z,p})p \end{aligned}$$

is a resolution of singularities, and the exceptional set

$$E = \varepsilon^{-1}(\Delta) \subset S^{[n]},$$

consisting of non-reduced subschemes, is an irreducible divisor on $S^{[n]}$. Let

$$\mathrm{pr}_i : S^n \to S, \ i = 1, \dots, n$$

and $\pi : S^n \to S^{(n)}$ denote the projections. Beauville [Bea83b, Prop. 6] showed that there is a natural injective map

$$j : H^2(S,\mathbb{Z}) \to H^2(S^{[n]},\mathbb{Z}),$$

such that $j(\alpha) = \varepsilon^*(\beta)$, where $\beta \in H^2(S^{(n)}, \mathbb{Z})$ satisfies $\pi^*(\beta) = \sum_i \mathrm{pr}_i^*(\alpha)$. The map j preserves the Hodge structure and the Beauville–Bogomolov form, and moreover one has

$$H^2(S^{[n]}, \mathbb{Z}) = j(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}e$$
$$\mathrm{NS}(S^{[n]}) = j(\mathrm{NS}(S)) \oplus \mathbb{Z}e,$$

where $2e = [E]$ is the class of the exceptional divisor. Furthermore, the exceptional divisor satisfies $(e, e) = 2 - 2n$, and thus

$$H^2(X, \mathbb{Z}) \cong L_n := L_{K3} \oplus \langle 2 - 2n \rangle = 3U \oplus 2E_8 \oplus \langle 2 - 2n \rangle$$

for any manifold X of $K3^{[n]}$-type. In particular, this implies $b_2(X) = 23$ and hence $h^{1,1}(X) = 21$.

1.3 Marked manifolds and the period map

The symplectic form ω defines an isomorphism $\omega : TX \xrightarrow{\sim} \Omega_X^1$. The fact that X is simply-connected implies $H^1(X, \mathcal{O}_X) = 0$ and hence

$$H^0(X, TX) \cong H^0(X, \Omega_X^1) = 0.$$

Thus the Kuranishi family

$$\pi : \mathcal{X} \to \mathrm{Def}(X)$$

is a universal deformation of X. We denote by $X_t := \pi^{-1}(t)$ its fibre over $t \in \mathrm{Def}(X)$. Since X_t is again an irreducible symplectic manifold, the number

$$h^{1,1}(X_t) = b_2(X_t) - 2$$

is constant, and therefore the Kuranishi family is universal for any of its fibres.

Theorem 1.3.1 (Bogomolov)**.** *The deformation space of X is unobstructed.*

Proof. [Bog78] □

This means that the deformation space $\mathrm{Def}(X)$ of X is a smooth germ of dimension $h^{1,1}(X) = b_2(X) - 2$.

We now consider manifolds X deformation equivalent to a given irreducible symplectic manifold X_0 and fix a lattice L such that $H^2(X_0, \mathbb{Z})$ is isometric to L.

Definition 1.3.2. A *marking* α of X is an isometry $\alpha : H^2(X, \mathbb{Z}) \to L$. The pair (X, α) is called a *marked manifold*. Two marked manifolds (X, α) and (X', α') are *isomorphic*, if there exists a biholomorphic map $f : X \to X'$ with $\alpha' = \alpha \circ f^*$, where $f^* : H^2(X', \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is the induced isometry.

Definition 1.3.3. For any lattice L, the corresponding *period domain* is given by

$$\Omega_L := \{\eta \in \mathbb{P}(L_{\mathbb{C}}) : (\eta, \eta) = 0 \text{ and } (\eta, \overline{\eta}) > 0\}.$$

Since the symplectic form of X satisfies $(\omega, \omega) = 0$ and $(\omega, \overline{\omega}) > 0$, the *period point*

$$P(X, \alpha) := \alpha(H^{2,0}(X)) \in \Omega_L$$

of a marked manifold (X, α) is a point in the period domain.

Let $\pi : \mathcal{X} \to S$ be a deformation of $X = \pi^{-1}(0)$ and $\alpha : H^2(X, \mathbb{Z}) \to L$ be a marking. If $U \subset S$ is a contractible open neighbourhood of 0, then α extends uniquely to a trivialization

$$\alpha_U : (R^2\pi_*\mathbb{Z})|_U \to L_U,$$

where L_U is the constant local system of stalk L on U.

Theorem 1.3.4 (Local Torelli)**.** *Let $\pi : \mathcal{X} \to \operatorname{Def}(X)$ be the Kuranishi family of X and $\alpha : H^2(X, \mathbb{Z}) \to L$ be a marking. The period map*

$$\begin{array}{rccl} P_\alpha : & \operatorname{Def}(X) & \to & \Omega_L, \\ & t & \mapsto & P(X_t, \alpha_t) \end{array}$$

is a local isomorphism.

Proof. [Bea83b, Thm. 5] □

Since $H^1(X, \mathcal{O}_X) = 0$, the map $c_1 : \operatorname{Pic}(X) \to \operatorname{NS}(X)$ is an isomorphism. Thus for a non-trivial line bundle $\mathcal{L} \in \operatorname{Pic}(X)$ and a marking $\alpha : H^2(X, \mathbb{Z}) \to L$ we have $0 \neq h := \alpha(c_1(\mathcal{L}))$. Let

$$\Omega_{h^\perp} := \{\eta \in \Omega_L : (\eta, h) = 0\}$$

be the set of period points orthogonal to h.

Corollary 1.3.5. *Let $\operatorname{Def}(X, \mathcal{L}) := P_\alpha^{-1}(\Omega_{h^\perp})$ and $\pi : \mathcal{X}_h \to \operatorname{Def}(X, \mathcal{L})$ be the restriction of the Kuranishi family. There exists a unique line bundle $\mathfrak{L}$ on $\mathcal{X}_h$ such that $\mathfrak{L}|_X = \mathcal{L}$. The family $(\mathcal{X}_h, \mathfrak{L})$ is a universal deformation of $(X, \mathcal{L})$.*

Proof. [Bea83b, Cor. 1] □

We denote by

$$\mathfrak{M}_L := \{(X, \alpha) : \alpha : H^2(X, \mathbb{Z}) \to L \text{ is a marking}\}/\cong$$

the moduli space of marked pairs. The following Proposition is a consequence of the Local Torelli theorem. A proof is given in [Huy12, Prop. 4.3].

Proposition 1.3.6. *The moduli space $\mathfrak{M}_L$ has the structure of a smooth analytic space of dimension $r(L) - 2$. For any $(X, \alpha) \in \mathfrak{M}_L$, there exists a natural holomorphic map* $\mathrm{Def}(X) \hookrightarrow \mathfrak{M}_L$ *identifying* $\mathrm{Def}(X)$ *with an open neighbourhood of (X, α) in $\mathfrak{M}_L$.*

The Local Torelli theorem now states that the period map $P : \mathfrak{M}_L \to \Omega_L$ is a local isomorphism.

Theorem 1.3.7 (Huybrechts)**.** *For any connected component $\mathfrak{M}_L^0 \subset \mathfrak{M}_L$, the restriction of the period map $P_0 : \mathfrak{M}_L^0 \to \Omega_L$ is surjective.*

Proof. [Huy99, Thm. 8.1] □

1.4 Global Torelli

By the Global Torelli theorem two K3 surfaces S and S' are isomorphic if and only if there exists an isomorphism $H^2(S, \mathbb{Z}) \to H^2(S', \mathbb{Z})$ preserving both the Hodge structure and the intersection form. The following theorem is a generalization for irreducible symplectic manifolds which was proved by Verbitsky.

Theorem 1.4.1 (Global Torelli)**.** *Let $\mathfrak{M}_L^0 \subset \mathfrak{M}_L$ be a connected component and $P_0 : \mathfrak{M}_L^0 \to \Omega_L$ the restriction of the period map.*

(i) The fiber $P_0^{-1}(\eta)$ consists of pairwise inseparable points for every $\eta \in \Omega_L$.

(ii) If (X_1, α_1) and (X_2, α_2) are two inseparable points of $\mathfrak{M}_L$, then X_1 and X_2 are bimeromorphic.

Proof. (i) is [Ver13, Thm. 1.18] and (ii) is [Huy99, Thm. 4.3]. This formulation of the theorem is given in [Mar11, Thm. 2.2]. □

The purpose of this section is to present some of its consequences, which are mainly due to Markman. Most results of this section can be found in Markman's survey article [Mar11].

1.4.1 Monodromy operators

The moduli space of marked K3 surfaces consists of two connected components, which are exchanged by the map $(X, \alpha) \mapsto (X, -\alpha)$. For irreducible symplectic manifolds of a given deformation type, this need not be true.

Definition 1.4.2. Let X_1, X_2 be irreducible symplectic manifolds.

(i) An isomorphism $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ is called a *parallel transport operator*, if there exists a smooth and proper family $\pi : \mathcal{X} \to S$ over an analytic base S, two base points $t_1, t_2 \in S$ with $\pi^{-1}(t_i) = X_i$ and a continuous path $\gamma : [0, 1] \to S$ with $\gamma(0) = t_1,\ \gamma(1) = t_2$, such that the parallel transport in $R^2\pi_*\mathbb{Z}$ along γ induces g.

(ii) An automorphism $H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ is called a *monodromy operator* of X, if it is a parallel transport operator . The set of monodromy operators of X is denoted by $\mathrm{Mon}^2(X)$.

As noted before, every parallel transport operator is an isometry with respect to the Beauville–Bogomolov form. Furthermore, the composition of parallel transport operators is again a parallel transport operator [Mar11, Footnote 3]. In particular, $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$ is a subgroup.

Theorem 1.4.3. $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$ *is a finite index subgroup.*

Proof. [Sul77], see also [Mar11, Lemma 7.5]. □

The isometry group $O(L)$ acts on $\mathfrak{M}_L$ by $\sigma(X,\alpha) = (X, \sigma\circ\alpha)$. For any connected component $\mathfrak{M}_L^0$ of the moduli space of marked pairs, the subgroup

$$\mathrm{Mon}(\mathfrak{M}_L^0) := \alpha \circ \mathrm{Mon}^2(X) \circ \alpha^{-1} \subset O(L)$$

is independent of the choice of $(X,\alpha) \in \mathfrak{M}_L^0$. By definition of monodromy operators and the universal property of $\mathfrak{M}_L$, the group $\mathrm{Mon}(\mathfrak{M}_L^0)$ is the subgroup of $O(L)$ fixing the connected component $\mathfrak{M}_L^0$ ([Mar11, Lemma 7.5]). A priori, the group $\mathrm{Mon}(\mathfrak{M}_L^0)$ depends on the choice of $\mathfrak{M}_L^0$. However, if $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$ is a normal subgroup, then the subgroup $\mathrm{Mon}(\mathfrak{M}_L^0) \subset O(L)$ is the same for every connected component.

Let X be a manifold of $K3^{[n]}$-type and $u \in H^2(X,\mathbb{Z})$ a class with $(u,u) \neq 0$. The reflection $R_u : H^2(X,\mathbb{Q}) \to H^2(X,\mathbb{Q})$ is given by

$$R_u(x) = x - \frac{2(x,u)}{(u,u)}u.$$

If $(u,u) = 2$ or $(u,u) = -2$, then R_u is an integral isometry. Moreover, let

$$\rho_u := \begin{cases} R_u & \text{if } (u,u) < 0 \\ -R_u & \text{if } (u,u) > 0. \end{cases}$$

Theorem 1.4.4 (Markman)**.** *For any manifold X of $K3^{[n]}$-type, the monodromy group is given by*

$$\mathrm{Mon}^2(X) = \langle \rho_u : \ u \in H^2(X,\mathbb{Z}) \text{ and } (u,u) = -2 \text{ or } (u,u) = 2\rangle$$

In particular, $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$ *is a normal subgroup.*

Proof. [Mar10, Thm. 1.2]. □

Therefore, one obtains a well-defined normal subgroup $\mathrm{Mon}(L_n) \subset O(L_n)$ by conjugation.

Using the concept of parallel transport operators, Markman obtained another formulation of the Global Torelli theorem. Let $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ be an isometry with respect to the Beauville–Bogomolov form. It is called a *Hodge isometry*, if it preserves the Hodge structures, that is, the Hodge decompositions of

$$H^2(X_i, \mathbb{C}) = H^2(X_i, \mathbb{Z}) \otimes \mathbb{C}.$$

Note that this is equivalent to $g(H^{2,0}(X_1)) = H^{2,0}(X_2)$. We denote by

$$\mathrm{Mon}^2_{\mathrm{Hdg}}(X) \subset \mathrm{Mon}^2(X)$$

the subgroup of monodromy operators which are Hodge isometries.

Theorem 1.4.5 (Hodge-theoretic Global Torelli)**.** *Let X_1, X_2 be irreducible symplectic manifolds and $g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z})$ a Hodge isometry which is a parallel transport operator.*

(i) The manifolds X_1 and X_2 are bimeromorphic.

(ii) If g maps some Kähler class to a Kähler class, then there exists a biholomorphic map $f : X_2 \to X_1$ with $f^* = g$.

Proof. [Mar11, Thm. 1.3] □

1.4.2 Orientation

Let L be a lattice of signature $(3, r(L) - 3)$, and for a period point $\eta \in \Omega_L$ let

$$L^{1,1}(\eta, \mathbb{R}) := \{x \in L_{\mathbb{R}} : (x, \eta) = 0\}.$$

Note that for any marked pair (X, α) with $P(X, \alpha) = \eta$, we have

$$\alpha(H^{1,1}(X, \mathbb{R})) = L^{1,1}(\eta, \mathbb{R}).$$

The positive cone

$$\mathcal{C}'_\eta := \{x \in L^{1,1}(\eta, \mathbb{R}) : (x, x) > 0\} \tag{1.1}$$

of $L^{1,1}(\eta, \mathbb{R})$ consists of two connected components.

On the other hand, let $h \in L$ be an element with $(h, h) > 0$. Since

$$\mathrm{sign}(h^\perp) = (2, r(L) - 3),$$

the hyperplane section

$$\Omega_{h^\perp} = \Omega_L \cap h^\perp \tag{1.2}$$

consists of two connected components.

We summarize [Mar11, Section 4], which describes how the choice of a connected component $\mathfrak{M}^0_L \subset \mathfrak{M}_L$ determines connected components of (1.1) and (1.2). This is done by showing that in both cases the connected components correspond to orientations (as defined below) of

$$\widetilde{\mathcal{C}}_L := \{x \in L_\mathbb{R} : (x,x) > 0\},$$

and that moreover any component of $\mathfrak{M}_L^0$ determines an orientation of $\widetilde{\mathcal{C}}_L$.

Lemma 1.4.6. *Let $W \subset L_\mathbb{R}$ be a three dimensional positive definite subspace. Then $W \setminus \{0\}$ is a deformation retract of $\widetilde{\mathcal{C}}_L$. In particular, $H^2(\widetilde{\mathcal{C}}_L, \mathbb{Z})$ is a free abelian group of rank 1.*

Proof. [Mar11, Lemma 4.1] □

A choice of *orientation* of $\widetilde{\mathcal{C}}_L$ is given by a generator of $H^2(\widetilde{\mathcal{C}}_L, \mathbb{Z})$. The homomorphism

$$O(L) \to \mathrm{Aut}(H^2(\widetilde{\mathcal{C}}_L, \mathbb{Z})) \cong \{1, -1\}$$

is the *real spinor norm.* The subgroup of isometries of spinor norm 1 is denoted by $O^+(L) \subset O(L)$.

Suppose that $\sigma \in O(L)$ is an isometry, such that there exists a positive three-dimensional subspace $W \subset L_\mathbb{R}$ with $\sigma(W) = W$. As a consequence of Lemma 1.4.6, we have $\sigma \in O^+(L)$ if and only if $\sigma|_W$ is orientation preserving.

We now fix an element $h \in L$ with $(h,h) > 0$. Then a point $\eta = \mathbb{C}\omega \in \Omega_{h^\perp}$ determines the positive definite space

$$\mathrm{Re}(\eta) \oplus \mathrm{Im}(\eta) \oplus \mathbb{R}h \subset L_\mathbb{R} \tag{1.3}$$

together with an oriented basis

$$(\mathrm{Re}(\omega),\, \mathrm{Im}(\omega),\, h). \tag{1.4}$$

This defines an orientation of $\widetilde{\mathcal{C}}_L$, which only depends on the connected component of $\Omega_{h^\perp}$ containing η.

On the other hand, if $\eta \in \Omega_L$ is fixed, then for any $h \in \widetilde{\mathcal{C}}_\eta$, (1.3) and (1.4) define an orientation of $\widetilde{\mathcal{C}}_L$ which only depends on the connected component of $\widetilde{\mathcal{C}}_\eta$ containing h.

Finally, let X be an irreducible symplectic manifold. The *positive cone* $\mathcal{C}_X$ of X is the connected component of

$$\mathcal{C}'_X := \{x \in H^{1,1}(X, \mathbb{R}) : (x,x) > 0\}$$

which contains the Kähler cone $\mathcal{K}_X$ of X. Therefore,

$$\widetilde{\mathcal{C}}_X := \{x \in H^2(X, \mathbb{R}) : (x,x) > 0\}$$

has a natural orientation, and if $\alpha : H^2(X, \mathbb{Z}) \to L$ is a marking, then the isomorphism $\alpha : \widetilde{\mathcal{C}}_X \to \widetilde{\mathcal{C}}_L$ defines an orientation of $\widetilde{\mathcal{C}}_L$ which only depends on the connected component $\mathfrak{M}_L^0$ containing (X, α). In particular, we have $\mathrm{Mon}(X) \subset O^+(H^2(X,\mathbb{Z}))$ and hence $\mathrm{Mon}(\mathfrak{M}_L^0) \subset O^+(L)$. Moreover, for any $h \in L$ with $(h,h) > 0$, there is a distinguished component $\Omega^+_{h^\perp}$ of $\Omega_{h^\perp}$ such that $\alpha^{-1}(h) \in \mathcal{C}_X$ for any $(X,\alpha) \in \mathfrak{M}_L^0$ with $P_0(X,\alpha) \in \Omega^+_{h^\perp}$.

We can now state another lattice-theoretic characterization of the monodromy group for $K3^{[n]}$-type manifolds given by Markman. An isometry $\sigma \in O(L_n)$ acts naturally on the discriminant group L_n^*/L_n. This defines a homomorphism

$$\pi : O^+(L_n) \to O(L_n^*/L_n).$$

For details, we refer to the next chapter.

Lemma 1.4.7. *The group* $\mathrm{Mon}(L_n)$ *is equal to the inverse image via* π *of the subgroup* $\{1,-1\} \subset O(L_n^*/L_n)$. *In particular, if* X *is a manifold of* $K3^{[n]}$*-type, then*

$$\mathrm{Mon}^2(X) = O^+(H^2(X,\mathbb{Z}))$$

if and only if $n = 2$ *or* $n-1$ *is a prime power.*

Proof. [Mar10, Lemma 4.2] □

1.4.3 Decomposition of the positive cone

Definition 1.4.8. Let X be an irreducible symplectic manifold.

(i) A *prime exceptional divisor* on X is an irreducible reduced effective divisor E with $(E,E) < 0$. We denote the set of classes of prime exceptional divisors on X by $\mathcal{P}_X \subset H^{1,1}(X,\mathbb{Z})$.

(ii) The *fundamental exceptional chamber* of $\mathcal{C}_X$ is the cone

$$\mathcal{FE}_X = \{x \in \mathcal{C}_X : (x,E) > 0 \text{ for every } E \in \mathcal{P}_X\}$$

(iii) An *exceptional chamber* of $\mathcal{C}_X$ is a subset of the form $g(\mathcal{FE}_X)$ for some isometry $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$.

Note that for K3 surfaces, prime exceptional divisors are the same as smooth rational curves, and the fundamental exceptional chamber is the Kähler cone. In higher dimensions, however, there is a decomposition of $\mathcal{FE}_X$ into chambers corresponding to bimeromorphic models of X, which we will describe now.

Proposition 1.4.9. *Let* $f : X \dashrightarrow Y$ *be a bimeromorphic map of irreducible symplectic manifolds.*

(i) f *is an isomorphism in codimension 1 and the induced map*

$$f^* : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$$

is a Hodge isometry.

(ii) f^* is a parallel transport operator.

Proof. (i) is due to O'Grady [O'G97, Prop. 1.6.2] and (ii) was shown by Huybrechts [Huy03, Cor. 2.7], as explained by Markman [Mar11, Thm. 3.1]. □

Proposition 1.4.10. *Let $g : H^2(Y,\mathbb{Z}) \to H^2(X,\mathbb{Z})$ be a parallel transport operator and a Hodge isometry. Then $g = f^*$ for some bimeromorphic map $f : X \dashrightarrow Y$ if and only if $g(\mathcal{FE}_Y) = \mathcal{FE}_X$.*

Proof. [Mar11, Cor. 5.7 and Lemma 5.12] □

Definition 1.4.11. The *birational Kähler cone* $\mathcal{BK}_X$ of X is the union of the cones $f^*\mathcal{K}_Y$ for all bimeromorphic maps $f : X \dashrightarrow Y$.

Since $(x, E) > 0$ for any Kähler class x and any effective class E, we have $\mathcal{K}_X \subset \mathcal{FE}_X$. Together with Proposition 1.4.10, this shows $\mathcal{BK}_X \subset \mathcal{FE}_X$. The decomposition of $\mathcal{FE}_X$ into the cones $f^*\mathcal{K}_Y$ of bimeromorphic models can be extended to all exceptional chambers:

Definition 1.4.12. A *Kähler-type chamber* of $\mathcal{C}_X$ is a subset of the form $g(f^*(\mathcal{K}_Y))$ for some isometry $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ and some bimeromorphic map $f : X \dashrightarrow Y$. The set of Kähler-type chambers of $\mathcal{C}_X$ is denoted by $\mathrm{KT}(X)$.

By definition, $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ acts on the Kähler-type chambers of X. On the other hand, $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ acts on the fibre $P_0^{-1}(P_0(X,\alpha))$ by

$$g(\widetilde{X},\widetilde{\alpha}) = (\widetilde{X}, \alpha \circ g \circ \alpha^{-1} \circ \widetilde{\alpha}).$$

Proposition 1.4.13. *Let $(X,\alpha) \in \mathfrak{M}^0_L$ be a marked pair. The map*

$$\begin{aligned} \rho : P_0^{-1}(P_0(X,\alpha)) &\to \mathrm{KT}(X) \\ (\widetilde{X},\widetilde{\alpha}) &\mapsto \alpha^{-1}(\widetilde{\alpha}(\mathcal{K}_{\widetilde{X}})) \end{aligned}$$

is a $\mathrm{Mon}^2_{\mathrm{Hdg}}(X)$-equivariant bijection.

Proof. [Mar11, Prop. 5.14] □

Proposition 1.4.13 can also be formulated in the following way. Let $\eta \in \Omega_L$ be a period point and $\mathcal{C}_\eta \subset \mathcal{C}'_\eta$ the connected component determined by $\mathfrak{M}^0_L$. A *Kähler-type chamber* of $\mathcal{C}_\eta$ is a subset of the form $\alpha(\mathcal{K})$, where $(X,\alpha) \in P_0^{-1}(\eta)$ and $\mathcal{K} \in \mathrm{KT}(X)$. The set of Kähler-type chambers of $\mathcal{C}_\eta$ is denoted by $\mathrm{KT}(\eta)$. Let

$$\mathrm{Mon}^2_{\mathrm{Hdg}}(\eta) := \{\sigma \in \mathrm{Mon}(\mathfrak{M}^0_L) : \sigma(\eta) = \eta\} \subset O(L).$$

Theorem 1.4.14. *The map*

$$\rho : P_0^{-1}(\eta) \to \mathrm{KT}(\eta)$$

given by $\rho(X,\alpha) = \alpha(\mathcal{K}_X)$ is a $\mathrm{Mon}^2_{\mathrm{Hdg}}(\eta)$-equivariant bijection.

Proof. [Mar11, Thm. 5.16] □

Chapter 2

Lattice theory

As we have seen, the Beauville–Bogomolov form defines a natural lattice structure on $H^2(X, \mathbb{Z})$, which together with the Hodge structure contains important information about X. The purpose of this chapter is to recall results on lattice theory that we will need later, mainly from Nikulin [Nik80b].

Definitions. Recall that a lattice is a finitely generated free abelian group L together with a non-degenerate bilinear form $(\, , \,) : L \times L \to \mathbb{Z}$. The lattice is called *even*, if $(v, v) \in 2\mathbb{Z}$ for every $v \in L$. The *discriminant* $\operatorname{discr} L$ is the determinant of the Gram matrix (e_i, e_j) with respect to some $\mathbb{Z}$-basis $\{e_i\}$ of L. The lattice L is unimodular if and only if $\operatorname{discr} L = \pm 1$.

We denote the signature of a lattice by $(l_{(+)}, l_{(-)})$. The lattice L is called *hyperbolic*, if $l_{(+)} = 1$. For any $0 \neq k \in \mathbb{Z}$, the lattice $L(k)$ is obtained by multiplying the bilinear form of L by k.

2.1 Finite quadratic forms

Definition 2.1.1. A *finite quadratic form* is a finite abelian group A together with a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ satisfying

(i) $q(na) = n^2 q(a)$ for all $n \in \mathbb{Z}$ and $a \in A$,

(ii) $q(a + a') - q(a) - q(a') \equiv 2b(a, a') \pmod{2\mathbb{Z}}$,

where $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ is a symmetric bilinear form. The form q is called *non-degenerate*, if the bilinear form b is non-degenerate.

The *isometry group* $O(A)$ is the group of automorphisms of A preserving the form q. For a subgroup $H \subset A$, we denote by $H^\perp \subset A$ its orthogonal complement.

Proposition 2.1.2. *Let $H \subset A$ be a subgroup.*

(i) If q is non-degenerate, then

$$|A| = |H||H^\perp|.$$

(ii) If $q|_H$ is non-degenerate, then

$$A = H \oplus H^\perp.$$

(iii) A finite quadratic form A splits orthogonally into its Sylow p-subgroups

$$A_p \subset A.$$

Proof. [Nik80b, Prop. 1.2.1 and Prop. 1.2.2] □

The *length* $l(A)$ of A is the minimal number of generators of the group A. We have $l(A) = \max_p l(A_p)$.

For the rest of this chapter, we only consider even lattices L. Since the bilinear form of L is non-degenerate, the map $v \mapsto (v, \cdot)$ defines an embedding $L \hookrightarrow L^*$ as a finite index subgroup and an isomorphism $L_{\mathbb{Q}} \cong L^*_{\mathbb{Q}}$. Therefore, there is a $\mathbb{Q}$-valued bilinear form on the *dual lattice*

$$L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z}) \subset L_{\mathbb{Q}}$$

and hence a non-degenerate quadratic form $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$ on the *discriminant group* $A_L := L^*/L$. The form q_L is called the *discriminant form* of L. An isometry $\varphi \in O(L)$ induces an isometry $\overline{\varphi} \in O(A_L)$. This defines a homomorphism

$$O(L) \to O(A_L).$$

Two lattices L_1, L_2 are called *stably equivalent*, if there exist unimodular lattices U_1, U_2 with $L_1 \oplus U_1 \cong L_2 \oplus U_2$.

Proposition 2.1.3. *The map $L \mapsto q_L$ defines a semi-group isomorphism between lattices up to stable equivalence and non-degenerate finite quadratic forms up to isomorphism.*

Proof. [Nik80b, Thm. 1.3.2] □

Definition 2.1.4. The *signature* of a finite quadratic form q is given by

$$\operatorname{sign} q := [l_{(+)} - l_{(-)}] \in \mathbb{Z}/8\mathbb{Z},$$

where L is a lattice with signature $(l_{(+)}, l_{(-)})$ and discriminant form q.

This is well-defined, since $u_{(+)} - u_{(-)} \equiv 0 \pmod 8$ for any unimodular lattice of signature $(u_{(+)}, u_{(-)})$ by [Nik80b, Thm. 1.1.1].

For any prime number p, a *p-adic lattice* and its discriminant form are defined in the same way, replacing $\mathbb{Z}$ by the p-adic integers $\mathbb{Z}_p$ and $\mathbb{Q}$ by the p-adic numbers $\mathbb{Q}_p$. Finite quadratic forms over $\mathbb{Z}_p$ can be identified with quadratic forms over $\mathbb{Z}$ which are defined on a finite abelian p-group [Nik80b, §1.7]. The discriminant $\operatorname{discr} L_p$ of a p-adic lattice is well-defined up to multiplication with $(\mathbb{Z}_p^*)^2$. The *genus* of a lattice L is given by the isometry classes of the lattices $L_p := L \otimes \mathbb{Z}_p$ and of $L_\infty := L \otimes \mathbb{R}$. Two lattices belong to the same genus if and only if their signatures are equal and their discriminant forms are isomorphic [Nik80b, Cor. 1.9.4].

2.2 Orthogonal extensions

In this section, we recall the results from [Nik80b, §1.4-1.5]. A sublattice $S \subset L$ is *primitive*, if L/S is a free group. Two primitive sublattices $S \subset L$ and $S' \subset L'$ are *isometric*, if there exists an isometry $\varphi : L \to L'$ with $\varphi(S) = S'$.

Let $S \subset L$ be a primitive sublattice and $K := S^\perp \subset L$ its orthogonal complement. The sequence of inclusions

$$S \oplus K \subset L \subset L^* \subset S^* \oplus K^*$$

defines an inclusion $H_L := L/(S \oplus K) \subset A_S \oplus A_K$ as an isotropic subgroup with

$$H_L^\perp / H_L \cong A_L.$$

Since discriminant forms are non-degenerate, this implies

$$|A_L| = \frac{|A_S||A_K|}{|H_L|^2}. \tag{2.1}$$

The restricted projections

$$p_S : H_L \to H_S := p_S(H_L) \quad \text{and} \quad p_K : H_L \to H_K := p_K(H_L)$$

are isomorphisms of groups, and the isomorphism

$$\gamma := p_K \circ p_S^{-1} : H_S \to H_K$$

is an anti-isometry.

Now consider another primitive sublattice $S' \subset L$ with orthogonal complement K' and let $\gamma' : H_{S'} \to H_{K'}$ be as above.

Proposition 2.2.1. *Let $\varphi : S \to S'$ and $\psi : K \to K'$ be isometries. The isometry*

$$\varphi \oplus \psi : S \oplus K \to S' \oplus K'$$

extends to an isometry of L if and only if $\overline{\psi} \circ \gamma = \gamma' \circ \overline{\varphi}$.

Proof. [Nik80b, Cor. 1.5.2] □

2.3 Existence and uniqueness

A finite quadratic form A is called *2-elementary* if $A \cong (\mathbb{Z}/2\mathbb{Z})^a$ as groups. The *parity* of A is given by

$$\delta(A) := \begin{cases} 0 & \text{if } q(a) \in \mathbb{Z}/2\mathbb{Z} \text{ for every } a \in A, \\ 1 & \text{else.} \end{cases}$$

Theorem 2.3.1. *A 2-elementary finite quadratic form is determined by its signature, its length, and its parity.*

Proof. [Nik80b, Thm. 3.6.2] □

Proposition 2.3.2. *The semi-group of non-degenerate 2-elementary finite quadratic forms is generated by the following forms:*

(i) $q_+(2)$*, the discriminant form of* $\langle 2\rangle$*,*

(ii) $q_-(2)$*, the discrminant form of* $\langle -2\rangle$*,*

(iii) $u(2)$*, the discriminant form of the (2-adic) lattice* $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$*,*

(iv) $v(2)$*, the discriminant form of the 2-adic lattice* $\begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}$*.*

Proof. This is a special case of [Nik80b, Prop. 1.8.1]. □

A	$\mathrm{sign}(A)$	$l(A)$	$\delta(A)$
$q_+(2)$	$1+8\mathbb{Z}$	1	1
$q_-(2)$	$-1+8\mathbb{Z}$	1	1
$u(2)$	$8\mathbb{Z}$	2	0
$v(2)$	$4+8\mathbb{Z}$	2	0

Table 2.1: Generators of 2-elementary finite quadratic forms

Theorem 2.3.3. *Let* A_p *be a quadratic form on a finite abelian* p*-group. There exists a* p*-adic lattice* $K(A_p)$ *of rank* $l(A_p)$ *with discriminant form isomorphic to* A_p*. It is unique, except in the case when* $p = 2$ *and* $A_2 \cong q_\pm(2) \oplus A_2'$ *for some finite quadratic form* A_2'*.*

Proof. [Nik80b, Thm. 1.9.1] □

Theorem 2.3.4. *A lattice of signature* $(l_{(+)}, l_{(-)})$ *with discriminant form* A *exists if and only if* *(i)* $l_{(+)}, l_{(-)} \geq 0$ *and* $l(A) \leq l_{(+)} + l_{(-)}$,

(ii) $\operatorname{sign}(A) \equiv l_{(+)} - l_{(-)} \pmod 8$,

(iii) $|A| \equiv (-1)^{l_{(-)}} \operatorname{discr} K(A_p) \pmod{(\mathbb{Z}_p^*)^2}$ *for all odd prime numbers* p *for which* $l(A_p) = l_{(+)} + l_{(-)}$,

(iv) $|A| \equiv \pm \operatorname{discr} K(A_2) \pmod{(\mathbb{Z}_2^*)^2}$ *if* $l(A_2) = l_{(+)} + l_{(-)}$ *and* A_2 *is not of the form* $q_\pm(2) \oplus A_2'$ *for some finite quadratic form* A_2'.

Proof. [Nik80b, Thm. 1.10.1] □

Theorem 2.3.5. *Let* L *be an indefinite lattice with discriminant group* A_L*, satisfying*

(i) $l((A_L)_p) \leq r(L) - 2$ *for all odd prime numbers* p*,*

(ii) if $l((A_L)_2) = r(L)$*, then* $(A_L)_2 \cong u(2) \oplus A'$ *or* $(A_L)_2 \cong v(2) \oplus A'$ *for some finite quadratic form* A'*.*

Then L *is unique in its genus and the homomorphism* $O(L) \to O(A_L)$ *is surjective.*

Proof. [Nik80b, Thm. 1.14.2] □

2.4 Orbits

We will frequently make use of the following lattice-theoretic fact.

Lemma 2.4.1. *Let* L *be an even lattice and* $k \in 2\mathbb{Z}$*. There are only finitely many* $O(L)$*-orbits of elements* $v \in L$ *with* $(v, v) = k$*.*

Proof. It is sufficient to show the claim for primitive elements $v \in L$. A primitive element $v \in L$ with $(v, v) = k$ is the same as a primitive embedding $S \hookrightarrow M$ where $S := \langle k \rangle$. For such an embedding, consider $H_L = L/(S \oplus S^\perp)$. Equation (2.1) gives

$$|A_{S^\perp}| = \frac{|A_L||H_L|^2}{k}.$$

Since $H_L \to A_S = \mathbb{Z}/k\mathbb{Z}$ is injective, there are only finitely many possibilites for $|A_{S^\perp}|$. By [Cas78, Ch. 9, Thm. 1.1], this implies that there are only finitely many possible isometry classes for $S^\perp$. On the other hand, it follows from Proposition 2.2.1 that for every lattice K, there are only finitely many isometry classes of embeddings $S \hookrightarrow L$ such that $S^\perp \cong K$. □

The *stable isometry group* $\widetilde{O}(L)$ of L is defined as

$$\widetilde{O}(L) := \{\sigma \in O(L) : \overline{\sigma} = \mathrm{id}_{A_L}\}.$$

The finiteness of A_L implies that $\widetilde{O}(L) \subset O(L)$ is a finite index subgroup.

Lemma 2.4.2. *Let $S \subset L$ be a sublattice and $\sigma \in \widetilde{O}(S)$. Then σ extends to an isometry in $\widetilde{O}(L)$ such that $\sigma|_{S^\perp} = \mathrm{id}_{S^\perp}$.*

Proof. [GHS13, Lemma 7.1] □

The *divisor* $\mathrm{div}_L(v)$, or simply $\mathrm{div}(v)$, of an element $v \in L$ is the positive generator of the ideal $(v, L) \subset \mathbb{Z}$. Equivalently, it is the unique positive integer, such that $v/\mathrm{div}(v)$ is a primitive element in L^*.

Proposition 2.4.3 (Eichler's criterion)**.** *Let L be an even lattice containing $U \oplus U$. The $\widetilde{O}(L)$ orbit of an element $v \in L$ is determined by (v, v) and $v/\mathrm{div}(v) \in L^*$.*

Proof. [GHS09, Prop. 3.3] □

Chapter 3

Automorphisms of order 3

In this chapter we will consider automorphisms of order 3 of irreducible symplectic fourfolds. For generalized Kummer varieties, automorphisms without fixed points have been described in [BNWS11] and [OS11], leading to a generalization of Enriques surfaces.

The only other known deformation type in dimension 4 is the $K3^{[2]}$-type. However, in this case, there are several explicit constructions of polarized families. One of them consists of Fano varieties $F(Y)$ of lines on cubic fourfolds Y. In the first section, we will consider examples of automorphisms of the Hilbert scheme $S^{[2]}$ that are induced by an automorphism of S, and automorphisms of $F(Y)$ that are induced by a polarized automorphism of Y.

The main part of this chapter is Section 2, where we will compute the Lefschetz formula for non-symplectic order 3 automorphisms, which relates the topology of the fixed locus to the action on the second cohomology group.

Definitions. An *automorphism* $f \in \mathrm{Aut}(X)$ is a biholomorphic map $f : X \to X$. If X is projective, then f is biregular. The *order* of an automorphism f is the order of the subgroup $\langle f \rangle \subset \mathrm{Aut}(X)$. An *involution* is an automorphism of order 2. The *fixed locus* of f is the set $X^f := \{x \in X : f(x) = x\}$.

Apart from its order, the main invariants of an automorphism $f : X \to X$ are its actions on the space $H^0(X, \Omega_X^2)$ and on the lattice $H^2(X, \mathbb{Z})$. The action on the 1-dimensional space $H^0(X, \Omega_X^2)$ is given by $f^*\omega = \lambda\omega$ for some $\lambda \in \mathbb{C}^*$. If moreover the order of f is a finite number d, then λ is a d-th root of unity.

Definition 3.0.4. The automorphism f is called *symplectic* if $f^*\omega = \omega$, and *non-symplectic* otherwise.

Recall that f induces an isometry $f^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ with respect to the Beauville–Bogomolov form.

Definition 3.0.5. The *invariant sublattice* of the automorphism f is given by

$$H^2(X, \mathbb{Z})^f = \{h \in H^2(X, \mathbb{Z}) : f^*(h) = h\}.$$

The *coinvariant sublattice* is the orthogonal complement of the invariant sublattice.

3.1 Examples

3.1.1 Natural automorphisms

One way to obtain automorphisms of irreducible symplectic manifolds is by starting with an automorphism $f : S \to S$ of a K3 surface S. This induces an automorphism of the Hilbert scheme of length n subschemes Z by

$$\begin{aligned} f^{[n]} : S^{[n]} &\to S^{[n]} \\ Z &\mapsto f(Z). \end{aligned}$$

Such an automorphism of $S^{[n]}$ is called *natural*. Clearly, f maps non-reduced subschemes to non-reduced subschemes, and thus leaves the exceptional divisor E globally invariant. Moreover, with respect to the natural embedding

$$j : H^2(S, \mathbb{Z}) \hookrightarrow H^2(S^{[n]}, \mathbb{Z}),$$

the restriction of $(f^{[n]})^*$ to $H^2(S, \mathbb{Z})$ is given by f^* (see [BS12, Section 3]). Therefore, the invariant lattice of $f^{[n]}$ is given by

$$H^2(S^{[n]}, \mathbb{Z})^{f^{[n]}} = j(H^2(S, \mathbb{Z})^f) \oplus \mathbb{Z}e.$$

The converse is also true:

Theorem 3.1.1 (Boissière–Sarti). *An automorphism of $S^{[n]}$ is natural if and only if it leaves the exceptional divisor globally invariant.*

Proof. [BS12, Thm. 1] □

In particular, any automorphism that fixes the divisor class $e \in H^2(S^{[n]}, \mathbb{Z})$ is natural, since E is rigid.

Now let us consider the action of $f^{[n]}$ on the symplectic form. Outside the exceptional divisor, the symplectic form ω of $S^{[n]}$ is induced by the $\mathfrak{S}_n$-symmetric symplectic form $\sum_{i=1}^n \mathrm{pr}_i^* \sigma$ on S^n, where $\sigma \in H^0(S, \Omega_S^2)$ is the symplectic form on the K3 surface S. Since the action of $f^{[n]}$ on ω is determined on this open subset, it is a symplectic automorphism if and only f is symplectic.

Example 3.1.2. Let S be a K3 surface and $i : S \to S$ an involution.

(i) If i is symplectic, then the fixed locus of $i^{[2]}$ consists of a K3 surface which is a smooth model of S/i, and 28 isolated points [Cam12].

(ii) If i is non-symplectic, then $\Gamma := S^i$ is a (not necessarily connected) curve. The fixed locus of $i^{[2]}$ consists of the surface $\Gamma^{(2)}$ and the quotient surface S/i [Bea11].

An example of an automorphism of the Hilbert scheme which is not natural was given by Beauville.

Example 3.1.3 (Beauville). Let $S \subset \mathbb{P}^3$ be a smooth quartic K3 surface not containing a line. For two generic points $p, q \in S$, the line $l = \overline{pq} \subset \mathbb{P}^3$ meets S in two further points r, s. The map

$$\begin{aligned} S^{[2]} &\dashrightarrow S^{[2]} \\ [p,q] &\mapsto [r,s] \end{aligned}$$

extends to a biregular involution of i of $S^{[2]}$. The fixed locus of i is isomorphic to the surface of bitangents of S, and therefore i is not induced by an involution of S.

3.1.2 Non-symplectic natural automorphisms of order 3

Let S be a $K3$ surface and $f : S \to S$ an automorphism of order 3 with $f^*\omega = \zeta\omega$, where $\zeta \in \mathbb{C}^*$ is a primitive third root of unity. We describe the fixed locus of the natural automorphism $f^{[2]} : S^{[2]} \to S^{[2]}$. This can also be found in [BCS14] for non-symplectic automorphisms of arbitrary prime order.

From now on we denote by $[s,t]$ the reduced subscheme of S supported at $\{s,t\}$. A reduced subscheme $[s,t] \in S^{[2]}$ is fixed by $f^{[2]}$ if and only if s and t are fixed by f. A non-reduced subscheme of length 2 supported on $s \in S$ is given by a tangent direction $\mathbb{C}v \in \mathbb{P}(T_sS)$. Such a point of $S^{[2]}$ is fixed by $f^{[2]}$ if and only if s is fixed by f and v is an eigenvector of df_s.

The fixed locus S^f on the K3 surface has been classified:

Theorem 3.1.4 (Artebani–Sarti). *The fixed locus S^f is the disjoint union of $n \leq 9$ points and $k \leq 6$ smooth curves with:*

(i) one curve of genus $g \geq 0$ and $k-1$ rational curves, or

(ii) $k = 0$ and $n = 3$.

Moreover, the rank of the coinvariant sublattice is an even number $2m$, and

$$m + n = 10, \qquad g = 3 + k - n.$$

Proof. [AS08, Thm. 2.2] □

Furthermore, df_s has eigenvalues $1, \zeta$, if s belongs to a fixed curve, and ζ^2, ζ^2, if s is an isolated fixed point [AS08, Section 2]. Thus the automorphism $f^{[2]}$ fixes exactly two non-reduced subschemes supported at s in the first case, and all of them in the second case.

Therefore, we have the following components in the fixed locus of $f^{[2]}$:

(i) The isolated point $[s,t]$ for every pair of distinct isolated fixed points $s,t \in S^f$.

(ii) The curve $\{[s,t] \in S^{[2]} : t \in C\} \cong C$ for every isolated point $s \in S^f$ and every curve $C \subset S^f$.

(iii) The surface $\{[s,t] \in S^{[2]} : s \in C_1, t \in C_2\} \cong C_1 \times C_2$ for every pair of distinct fixed curves $C_1, C_2 \subset S^f$.

(iv) The curve $\mathbb{P}(T_s S) \cong \mathbb{P}^1$ of non-reduced subschemes supported at s for every isolated fixed point $s \in S^f$.

(v) The curve $\mathbb{P}(N_\zeta) \cong C$, where $N_\zeta \subset TS|_C$ is the eigenbundle corresponding to the eigenvalue ζ of $df|_{TC}$, for every fixed surve $C \subset S^f$.

(vi) The surface $C^{(2)}$ which is the closure of $\{[s,t] \in S^{[2]} : s,t \in C,\ s \neq t\}$ for every fixed curve $C \subset S^f$. This surface meets the exceptional divisor in the curve $\mathbb{P}(TC) \cong C$.

Thus, in case (i) of Theorem 3.1.4, the fixed locus of $f^{[2]}$ consists of $\binom{n}{2}$ isolated fixed points, $nk+k-1$ rational curves, $n+1$ curves of genus g, $k-1$ projective planes, $\binom{k-1}{2}$ quadric surfaces, $k-1$ products $\mathbb{P}^1 \times C_g$ and the symmetric square $C_g^{(2)}$. In case (ii), it consists of 3 isolated points and 3 rational curves.

3.1.3 Fano varieties of cubic fourfolds

Another explicit construction of $K3^{[2]}$-type manifolds was given by Beauville and Donagi. Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold and

$$F(Y) := \{l \in \mathrm{Gr}(1,\mathbb{P}^5) : l \subset Y\}$$

be the Fano variety of lines on Y.

Proposition 3.1.5 (Beauville–Donagi)**.** *$F(Y)$ is an irreducible symplectic manifold of $K3^{[2]}$-type.*

Proof. [BD85, Prop. 1] □

This is shown by construction of an isomorphism $S^{[2]} \to F(Y)$, where Y is a special type of cubic, called a *Pfaffian cubic*, and S is a K3 surface related to Y.

For a six-dimensional complex vector space V, let $\Delta \subset \mathbb{P}(\Lambda^2 V)$ be the hypersurface of degenerate forms and $G \subset \Delta$ be the subvariety of forms of rank ≤ 2. In the same way, one defines subvarieties $G^* \subset \Delta^* \subset \mathbb{P}(\Lambda^2 V^*)$. Since the determinant of a skew-symmetric 6×6 matrix $A = (x_{i,j})$ is the square of a cubic polynomial, the variety

$$\Delta^* \subset \mathbb{P}(\Lambda^2 V^*) \cong \mathbb{P}^{14}$$

is a cubic hypersurface. Moreover, $G \cong \mathrm{Gr}(2, V)$ is the Grassmannian of planes in V.

Let $L \subset \mathbb{P}(\Lambda^2 V)$ be an 8-dimensional linear subspace and

$$\mathbb{P}^5 \cong L^* \subset \mathbb{P}(\Lambda^2 V^*)$$

be the dual subspace. Furthermore, let

$$S := G \cap L, \qquad Y := \Delta^* \cap L^*.$$

For the generic choice of L, the surface S is a K3 surface, and $Y \subset L^*$ is a smooth cubic fourfold. Cubic fourfolds that arise in this way are called *Pfaffian cubic fourfolds.*

Proposition 3.1.6. *If L is chosen sufficiently generic, such that Y and S are smooth, Y does not to contain a plane, and S does not to contain a line, then the varieties $F(Y)$ and $S^{[2]}$ are isomorphic.*

Proof. [BD85, Prop. 5] □

The isomorphism is constructed in the following way. Let $P, Q \in S$ be two distinct points, regarded as planes in V. Let $l \subset L^\perp$ be the subspace of forms vanishing on the 4-plane $P + Q$. Since all forms in $L^\perp$ vanish on P and Q, this is a linear subspace of dimension at least 1. On the other hand, every form in l is degenerate, which implies $l \subset \Delta^*$. By assumption, l is a line, and the map given by $[P, Q] \mapsto l$ extends to the exceptional divisor. The inverse morphism is also explicitly given in [BD85].

Automorphisms of Fano varieties. Let $Y \subset \mathbb{P}^5$ be a smooth cubic fourfold and $\sigma : \mathbb{P}^5 \to \mathbb{P}^5$ be an automorphism with $\sigma(Y) = Y$. Then σ induces an automorphism $g : F(Y) \to F(Y)$ of the Fano variety of lines. In this section, we describe the fixed locus for order 3 automorphisms obtained in this way. This can also be found in [Fu13] for the symplectic case, and in [BCS14] for the non-symplectic case.

Example 3.1.7. Since any automorphism $\sigma \in \mathrm{Aut}(\mathbb{P}^5)$ of order 3 is diagonalizable, one can assume that

$$\sigma(x_0 : \ldots : x_5) = (\zeta^{k_0} x_0 : \ldots : \zeta^{k_5} x_5).$$

We simply write $\sigma = (\zeta^{k_0}, \ldots, \zeta^{k_5})$. By [GAL11, Thm. 2.8], all smooth cubic fourfolds $Y = V(f_Y)$ with $\sigma^* f_Y = f_Y$ are given in the list below. By [BCS14, Lemma 6.2], in this case the induced automorphism of $F(Y)$ is symplectic if and only if $\det(\sigma) = 1$.

(i)

$$\begin{aligned} \sigma &= (\zeta, 1, 1, 1, 1, 1) \\ f_Y &= x_0^3 + f_3(x_1, \ldots, x_5) \end{aligned}$$

A line in Y is globally invariant if and only if it contains two fixed points. The fixed locus of σ consists of the isolated point $(1 : 0 : \ldots : 0)$, which does not belong to Y, and the hyperplane $H = \{x_0 = 0\}$. Hence the fixed locus of g is the Fano surface of lines on the smooth cubic threefold $H \cap Y$.

(ii)

$$\begin{aligned} \sigma &= (\zeta, \zeta, 1, 1, 1, 1) \\ f_Y &= f_3(x_0, x_1) + g_3(x_2, \ldots, x_5) \end{aligned}$$

The fixed locus of σ consists of a line $L = \{x_2 = x_3 = x_4 = x_5 = 0\}$ and a 3-plane $P = \{x_0 = x_1 = 0\}$. The intersections $Y \cap L$ and $Y \cap P$ are smooth, since Y is smooth. Therefore $Y \cap L = \{p_1, p_2, p_3\}$ consists of three distinct points and $S = Y \cap P$ is a smooth cubic surface. The line spanned by p_i and p_j, $i \neq j$ is never contained in Y. On the other hand, a line between p_i and any $q \in S$ is always contained in Y, as are the 27 lines on S. Thus the fixed locus of g consists of three cubic surfaces and 27 isolated points.

(iii)

$$\begin{aligned} \sigma &= (\zeta^2, \zeta, \zeta, 1, 1, 1) \\ f_Y &= x_0^3 + f_3(x_1, x_2) + g_3(x_3, x_4, x_5) \\ &\quad + x_0 x_1 l_1(x_3, x_4, x_5) + x_0 x_2 m_1(x_3, x_4, x_5) \end{aligned}$$

The fixed locus of σ consists of the point $(1 : 0 : \ldots : 0)$, the line

$$L = \{x_0 = x_3 = x_4 = x_5 = 0\},$$

and the plane $P = \{x_0 = x_1 = x_2 = 0\}$. As before, Y meets the line in three distinct points p_1, p_2, p_3 and the plane in an elliptic curve E. A line spanned by p_i and p_j is never contained in Y, while a line spanned by p_i and a point on E is always contained in Y. Hence the fixed locus of g consists of three copies of E.

(iv)

$$\begin{aligned} \sigma &= (\zeta^2, \zeta, 1, 1, 1, 1) \\ f_Y &= a_0 x_0^3 + a_1 x_1^3 + f_3(x_2, \ldots, x_5) \\ &\quad + x_0 x_1 l_1(x_2, \ldots, x_5), \qquad a_0, a_1 \neq 0 \end{aligned}$$

The fixed locus of σ consists of two isolated points which do not belong to Y and the 3-plane $P = \{x_0 = x_1 = 0\}$ which meets Y in a smooth cubic surface. The fixed locus of g consists of 27 points.

(v)

$$\begin{aligned} \sigma &= (\zeta,\zeta,\zeta,1,1,1) \\ f_Y &= f_3(x_0,x_1,x_2) + g_3(x_3,x_4,x_5) = 0 \end{aligned}$$

The fixed locus of σ consists of two planes which meet Y in elliptic curves E, F. A line spanned by two points of the same curve is not contained in Y, while a line spanned by a point on E and a point on F is. Thus the fixed locus of g is the abelian surface $E \times F$.

(vi)

$$\begin{aligned} \sigma &= (\zeta^2,\zeta^2,\zeta,\zeta,1,1) \\ f_Y &= f_3(x_0,x_1) + g_3(y_0,y_1) + h_3(z_0,z_1) \\ &\quad + \textstyle\sum_{i,j,k} x_i y_j z_k \end{aligned}$$

The fixed locus of σ consists of three lines L_1, L_2, L_3. Every line L_i meets Y in three points $p_{ij},\ j = 1,2,3$. The line spannend by p_{ij} and p_{kl} is contained in Y if and only if $i \neq k$. Thus g fixes 27 points.

Furthermore, there is exactly one family of smooth cubics $Y = V(f_Y)$ with $\sigma^* f_Y = \zeta f_Y$. In this case, one has $\sigma = (\zeta^2,\zeta^2,\zeta,\zeta,1,1)$. In [BCS14, Exm. 6.7], it is shown that the invariant lattice is the same as for the natural automorphism with $n = 3$ and $k = 0$. Moreover, the fixed locus is given by 3 rational curves and 3 isolated points and is therefore also the same as in the natural case.

Automorphisms of Pfaffian cubics. Assume that $\sigma \in \mathrm{Aut}(\mathbb{P}(\Lambda^2 V^*))$ is an automorphism with $\sigma(\Delta^*) = \Delta^*$ and that $L^* \subset \mathbb{P}(\Lambda^2 V^*)$ is a subspace such that $\sigma(L^*) = L^*$. Then σ induces an automorphism of the Pfaffian cubic $Y = \Delta^* \cap L^*$ and therefore on $F(Y)$. If L satisfies the assumptions of Proposition 3.1.5, then using the isomorphism $S^{[2]} \to F(Y)$, we obtain an automorphism of $f : S^{[2]} \to S^{[2]}$. If moreover the automorphism σ is induced by an automorphism of V, then this construction also induces an automorphism on S, and f is the corresponding natural automorphism. On the other hand, if this is not the case, one might obtain an example of a non-natural automorphism on $S^{[2]}$.

We now give an example which shows that smooth Pfaffian cubics with automorphisms of order 3 exist.

Example 3.1.8. After choosing a basis of V^*, elements of $\Lambda^2 V^*$ are represented by skew-symmetric 6×6 matrices. The automorphism $\sigma = (1,1,\zeta,\zeta,\zeta^2,\zeta^2)$ of V^* induces the automorphism

$$\begin{pmatrix} 0 & x_0 & x_1 & x_2 & x_3 & x_4 \\ & 0 & x_5 & x_6 & x_7 & x_8 \\ & & 0 & x_9 & x_{10} & x_{11} \\ & & & 0 & x_{12} & x_{13} \\ & & & & 0 & x_{14} \\ & & & & & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & x_0 & \zeta x_1 & \zeta x_2 & \zeta^2 x_3 & \zeta^2 x_4 \\ & 0 & \zeta x_5 & \zeta x_6 & \zeta^2 x_7 & \zeta^2 x_8 \\ & & 0 & \zeta^2 x_9 & x_{10} & x_{11} \\ & & & 0 & x_{12} & x_{13} \\ & & & & 0 & \zeta x_{14} \\ & & & & & 0 \end{pmatrix}$$

The fixed locus in $\mathbb{P}(\Lambda^2 V^*)$ is given by three linear subspaces $P_0, P_1, P_2 \cong \mathbb{P}^4$. We choose three lines $L_i \subset P_i$ and consider the subspace $L^* \cong \mathbb{P}^5$ spanned by these lines. Then $Y := \Delta^* \cap L^*$ is a Pfaffian cubic with $\sigma(Y) = Y$. Moreover, for the generic choice of L_0, L_1, L_2, the cubic Y is smooth. For example, one can choose

$$\begin{array}{llllll} L_0: & x_0 + x_{11} - x_{13} & = x_{10} - x_{11} + x_{12} & = x_{12} + x_{13} & = 0 \\ L_1: & x_1 + x_6 - x_{14} & = x_2 - x_6 + x_{14} & = x_5 & = 0 \\ L_2: & x_3 + x_4 + x_7 - x_8 & = x_3 - x_4 + x_7 - x_9 & = x_3 - x_7 + x_8 & = 0. \end{array}$$

The automorphism σ is of type (vi). However, we do not know whether the generic Pfaffian in this family contains a plane.

3.2 Lefschetz formula for order 3 automorphisms

If $f : X \to X$ is a symplectic automorphism, then the symplectic form on X restricts to a symplectic form on every component of the fixed locus. If $\dim X = 4$, this means that X^f is the union of isolated points, K3 surfaces and 2-dimensional tori. Moreover, if X is of $K3^{[2]}$-type, the fixed locus has been completely classified for automorphisms of prime order [Mon14, Cor. 5.2]. For involutions, the result is the following.

Theorem 3.2.1 (Camere, Mongardi)**.** *Assume that X is a $K3^{[2]}$-type manifold and $i : X \to X$ is a symplectic involution. The fixed locus X^i is the union of a K3 surface and 28 isolated points.*

Proof. [Cam12, Thm. 5] and [Mon12, Thm. 4.1] □

In the non-symplectic case, the fixed locus need not be symplectic. Non-symplectic involutions have been studied in [Bea11].

Theorem 3.2.2 (Beauville)**.** *Let X be a symplectic fourfold with $b_2(X) = 23$ and $i : X \to X$ a non-symplectic involution. Let t denote the trace of i^* acting on $H^{1,1}(X)$. The fixed locus of i is a (not necessarily connected) surface F with*

$$K_F^2 = t^2 - 1, \qquad \chi(\mathcal{O}_F) = \frac{1}{8}(t^2 + 7), \qquad e(F) = \frac{1}{2}(t^2 + 23).$$

Proof. [Bea11, Thm. 2] □

For both theorems, the holomorphic Lefschetz formula was used to obtain topological information about the fixed locus. In this section, we compute the Lefschetz formula for non-symplectic automorphisms of order 3 on $K3^{[2]}$-type fourfolds.

3.2.1 Holomorphic Lefschetz formula

Let Y be a compact complex manifold and V a complex vector bundle of rank r on Y. As a consequence of the splitting principle, there exist elements $x_1, \dots, x_r$ in some ring extension of $H^*(Y, \mathbb{Z})$, such that the total Chern class of V is given by

$$c(V) = \sum_{j=0}^{r} c_j(V) = \prod_{i=1}^{r} (1 + x_i),$$

that is, $c_j(V) = \sigma_j(x_1, \dots, x_r)$, where σ_j is the j-th elementary symmetric polynomial in r variables.

The Chern character $\mathrm{ch}(V)$ and the Todd class $\mathrm{Td}(V)$ of V are defined as

$$\begin{aligned} \mathrm{ch}(V) &= \sum_{i=1}^{r} e^{x_i}, \\ \mathrm{Td}(V) &= \prod_{i=1}^{r} \frac{x_i}{1 - e^{-x_i}}. \end{aligned}$$

For $\dim Y \leq 2$ we have

$$\begin{aligned} \mathrm{ch}(V) &= r + c_1(V) + \frac{1}{2} c_1^2(V) - c_2(V), \\ \mathrm{Td}(V) &= 1 + \frac{1}{2} c_1(V) + \frac{1}{12} (c_1^2(V) + c_2(V)). \end{aligned}$$

We now consider a holomorphic vector bundle V on Y with a holomorphic automorphism $f : V \to V$ of finite order acting trivially on Y. By [Ati67, Prop. 1.6.2], the bundle admits a splitting

$$V = \bigoplus_{\vartheta} V_{\vartheta},$$

where $V_{\vartheta} \subset V$ is the eigenbundle corresponding to the eigenvalue ϑ of f. Let

$$\begin{aligned} \mathrm{ch}(V, f) &:= \sum_{\vartheta} \vartheta \cdot \mathrm{ch}(V_{\vartheta}), \\ \mathrm{ch}\, \lambda_{-1}(V, f) &:= \sum_{i=0}^{r} (-1)^i \,\mathrm{ch}(\wedge^i V, f). \end{aligned}$$

Using the graded isomorphism $\wedge(V \oplus W) \cong \wedge(V) \otimes \wedge(W)$ and the identity $\mathrm{ch}(V \otimes W) = \mathrm{ch}(V)\,\mathrm{ch}(W)$, we obtain

$$\mathrm{ch}\, \lambda_{-1}(V, f) = \prod_{\vartheta} \mathrm{ch}\, \lambda_{-1}(V_{\vartheta}, f).$$

Now let X be a compact complex manifold and $f : X \to X$ a biholomorphic map of finite order. By [Car35], for every fixed point $p \in X^f$, there exists a local coordinate system around p which linearizes f. Therefore, the fixed locus X^f is a disjoint union of complex submanifolds Y, and we have a decomposition

$$TX|_Y = TY \oplus \bigoplus_{\vartheta \neq 1} N_\vartheta,$$

where $N_\vartheta \subset TX|_Y$ is the eigenbundle corresponding to ϑ. In particular, there is a natural isomorphism

$$N_{Y/X} \cong \bigoplus_{\vartheta \neq 1} N_\vartheta$$

and hence a natural action of f on $N_{Y/X}$ without eigenvalue 1. For any connected component $Y \subset X^f$ let

$$\nu(Y,V) := \int_Y \frac{\operatorname{ch}(V|_Y, f)\operatorname{Td}(Y)}{\operatorname{ch}\lambda_{-1}(N^*_{Y/X}, f)}.$$

The holomorphic Lefschetz number of f is defined as

$$L(f,V) := \sum_{i=0}^{n} (-1)^i \operatorname{tr} f|H^i(X, \mathcal{O}(V)).$$

Theorem 3.2.3 (Holomorphic Lefschetz Theorem)**.**

$$L(f,V) = \sum_{Y \subset X^f} \nu(Y,V).$$

Proof. [AS68, Thm. 3.9] □

3.2.2 Non-symplectic automorphisms of order 3

Let X be an irreducible symplectic fourfold and $f : X \to X$ an automorphism of order 3 with $f^*\omega = \zeta\,\omega$, where ζ is a primitive third root of unity.

Lemma 3.2.4. *For any fixed component $Y \subset X^f$, one of the following cases holds:*

(i) Y is an isolated point, $\operatorname{rk} N_\zeta = 0$ *and* $\operatorname{rk} N_{\zeta^2} = 4$*,*

(ii) Y is a smooth curve, $\operatorname{rk} N_\zeta = 1$ *and* $\operatorname{rk} N_{\zeta^2} = 2$*,*

(iii) Y is a smooth Lagrangian surface, $\operatorname{rk} N_\zeta = 2$ *and* $\operatorname{rk} N_{\zeta^2} = 0$*.*

Furthermore, for any connected component $Y \subset X^f$, the symplectic form of X restricts to an everywhere non-degenerate form on the eigenbundle $N_{\zeta^2} \subset N_{Y/X}$.

Proof. For any point $p \in Y$ and any pair of tangent vectors $v \in (N_{\zeta^2})_p$ and $w \in T_pY$, we have

$$\zeta\omega(v,w) = f^*\omega(v,w) = \omega(df_p(v), df_p(w)) = \zeta^2(df_p(v), df_p(w)),$$

which shows that N_{ζ^2} is orthogonal to TY. In the same way it follows that N_{ζ^2} is orthogonal to N_ζ. Therefore, the non-degeneracy of ω implies the non-degeneracy of $\omega|_{N_{\zeta^2}}$. In particular, the rank of N_{ζ^2} is even. The rest of the statement follows from the fact that $\det(df_p) = \zeta^2$ for every $p \in X^f$. $\square$

We denote by N the number of isolated fixed points of f. Let $C \subset X^f$ be the total fixed curve, $S \subset X^f$ the total fixed surface, and $C_i \subset C$ and $S_j \subset S$ their connected components. We write

$$\begin{aligned} \chi(C) &:= \sum_i \chi(C_i) = \sum_i (2 - 2g(C_i)) \\ c_1^2(S) &:= \sum_j c_1^2(S_j) \\ c_2(S) &:= \sum_j c_2(S_j). \end{aligned}$$

As in the case of symplectic involutions [Cam12], we apply the Lefschetz theorem to the bundles $V = \mathcal{O}_X, \Omega_X^1$ and Ω_X^2. We first compute the numbers $\nu(Y, V)$ for any irreducible symplectic fourfold. However, in the general case, we only compute one Lefschetz number $L(f, \mathcal{O}_X)$. In Proposition 3.2.6, we compute the other two Lefschetz numbers for $K3^{[2]}$-type manifolds.

Proposition 3.2.5. *Let X be an irreducible symplectic fourfold and $f : X \to X$ a non-symplectic automorphism of order 3. Then*

$$\begin{aligned} 2N - \chi(C) + \tfrac{1}{2}(3c_1^2(S) - 5c_2(S)) \quad &= 0 \\ 8N + 5\chi(C) - (3c_1^2(S) + c_2(S)) \quad &= 18\zeta^2 \cdot L(f, \Omega_X^1) \\ 2N + 2\chi(C) + \tfrac{1}{4}(3c_1^2(S) + 11c_2(S)) \quad &= 3 \cdot L(f, \Omega_X^2). \end{aligned}$$

Proof. (i) We apply the holomorphic Lefschetz formula to the trivial line bundle $V = \mathcal{O}_X$. Since

$$H^2(X, \mathcal{O}_X) = \overline{H^0(X, \Omega_X^2)} = \mathbb{C}\,\overline{\omega}$$

and

$$H^4(X, \mathcal{O}_X) = \overline{H^0(X, \Omega_X^4)} = \mathbb{C}\,\overline{\omega}^2,$$

the Lefschetz number of f is $L(f, \mathcal{O}_X) = 1 + \zeta^2 + \zeta = 0$.

Let first $Y = p$ be an isolated fixed point. Then

$$\mathrm{ch}(\wedge^i N_{\zeta^2}^*) = \dim(\wedge^i N_{\zeta^2}^*) = \binom{4}{i},$$

and we obtain

$$\nu(p, \mathcal{O}_X) = \frac{1}{1 - 4\zeta^2 + 6\zeta - 4 + \zeta^2} = (1 - \zeta^2)^{-4} = \frac{\zeta^2}{9}.$$

Now let $Y = C$ be a curve. Since N_{ζ^2} is a holomorphic symplectic bundle, we have $\det N_{\zeta^2} \cong \mathcal{O}_C$ and therefore $c_1(N_{\zeta^2}) = c_1(\det N_{\zeta^2}) = 0$. From $c_1(TX|_C) = 0$, we obtain $c_1(N_\zeta) = -c_1(C)$.

$$\begin{aligned}
\nu(C, \mathcal{O}_X) &= \int_C \frac{\mathrm{Td}(C)}{(1-\zeta\cdot \mathrm{ch}(N_\zeta^*))(1-\zeta^2\cdot \mathrm{ch}(N_{\zeta^2}^*)+\zeta\cdot \mathrm{ch}(\det N_{\zeta^2}^*))} \\
&= \int_C \frac{1+c_1(C)/2}{(1-\zeta(1+c_1(C))(1-2\zeta^2+\zeta)} \\
&= \int_C \frac{1+c_1(C)/2}{(1-\zeta-\zeta c_1(C))(1-\zeta^2)^2} \\
&= \int_C \frac{(1+c_1(C)/2)(1-\zeta+\zeta c_1(C))}{(1-\zeta)^2(1-\zeta^2)^2} = \frac{1}{9}\int_C (1-\zeta-\frac{\zeta^2}{2}c_1(C)) \\
&= -\frac{\zeta^2}{18}\chi(C).
\end{aligned}$$

For a fixed surface $Y = S$ we have $N_{S/X} = N_\zeta$, and by [Bea11, Lemma 1.3], the fact that S is Lagrangian implies $N_{S/X}^* \cong TS$.

$$\begin{aligned}
\nu(S, \mathcal{O}_X) &= \int_S \frac{\mathrm{Td}(S)}{1-\zeta\cdot \mathrm{ch}(N_{S/X}^*)+\zeta^2\,\mathrm{ch}(\det N_{S/X}^*)} \\
&= \int_S \frac{1+c_1(S)/2+(c_1^2(S)+c_2(S))/12}{1-\zeta[2+c_1(S)+\frac{1}{2}c_1^2(S)-c_2(S)]+\zeta^2[1+c_1(S)+\frac{1}{2}c_1^2(S).]}
\end{aligned}$$

Computing this fraction in $\mathbb{C}[[x_1, x_2]]$, where $c_1(S) = x_1 + x_2$ and $c_2(S) = x_1x_2$, we obtain

$$\begin{aligned}
\nu(S) &= \frac{\zeta^2}{36}\int_S [3x_1^2 + x_1x_2 + 3x_2^2 + 2(\zeta^2-\zeta) - 12] \\
&= \frac{\zeta^2}{36}\int_S [3c_1^2(S) - 5c_2(S) + 2(\zeta^2-\zeta)c_1(S) - 12] \qquad (3.1)\\
&= \frac{\zeta^2}{36}(3c_1^2(S) - 5c_2(S)).
\end{aligned}$$

Thus the Lefschetz formula gives

$$2N - \chi(C) + \frac{1}{2}(3c_1^2(S) - 5c_2(S)) = 0. \qquad (3.2)$$

(ii) Next we apply the formula to the cotangent bundle Ω_X^1. If p is an isolated fixed point, then $\mathrm{ch}(\Omega_X^1|_p, f) = 4\zeta^2$ and

$$\nu(p, \Omega_X^1) = \frac{\zeta^2}{9}\cdot 4\zeta^2 = \frac{4}{9}\zeta.$$

If C is a fixed curve, then

$$\begin{aligned}\mathrm{ch}(\Omega^1_X|_C, f) &= \mathrm{ch}(\Omega^1_C) + \zeta\,\mathrm{ch}(N^*_\zeta) + \zeta^2\,\mathrm{ch}(N^*_{\zeta^2})\\ &= 1 - c_1(C) + \zeta(1 + c_1(C)) + 2\zeta^2.\end{aligned}$$

We get

$$\begin{aligned}\nu(C, \Omega^1_X) &= \int_Y (1 - c_1(C) + \zeta(1 + c_1(C)) + 2\zeta^2) \cdot \frac{1}{9}(1 - \zeta - \frac{\zeta^2}{2} c_1(C))\\ &= \frac{5}{18}\zeta \cdot \chi(C).\end{aligned}$$

For a fixed surface S, we have

$$\begin{aligned}\mathrm{ch}(\Omega^1_X|_S, f) &= \mathrm{ch}(\Omega^1_S) + \zeta\,\mathrm{ch}(TS)\\ &= 2 - c_1(S) + \frac{1}{2}c_1^2(S) - c_2(S) + \zeta(2 + c_1(S) + \frac{1}{2}c_1^2(S) - c_2(S)).\end{aligned}$$

Multiplying this with (3.1), we obtain

$$\nu(S, \Omega^1_X) = -\frac{1}{18}\zeta\,(3c_1^2(S) + c_2(S)).$$

Thus the Lefschetz formula gives

$$8N + 5\chi(C) - (3c_1^2(S) + c_2(S)) = 18\zeta^2 L(f, \Omega^1_X). \tag{3.3}$$

(iii) Now we apply the formula to the bundle Ω^2_X. If $Y = p$ is an isolated point, then

$$\mathrm{ch}(\Omega^2_X|_p, f) = \zeta \cdot \mathrm{rk}(\Omega^2_X) = 6\zeta$$

and thus

$$\nu(p, \Omega^2_X) = \frac{\zeta^2}{9} \cdot 6\zeta = \frac{2}{3}.$$

For a curve $Y = C$, we have

$$\Omega^2_X|_C = (\Omega^1_C \otimes N^*_\zeta) \oplus (\Omega^1_C \otimes N^*_{\zeta^2}) \oplus (N^*_\zeta \otimes N^*_{\zeta^2}) \oplus \Lambda^2 N^*_{\zeta^2}.$$

Using $\Lambda^2 N^*_{\zeta^2} \cong \mathcal{O}_C$, we obtain

$$\begin{aligned}\mathrm{ch}(\Omega^2_X|_C, f) &= \zeta(1 - c_1(C))(1 + c_1(C)) + 2\zeta^2(1 - c_1(C)) + 2(1 + c_1(C)) + \zeta\\ &= 2(\zeta + 2)c_1(C)\end{aligned}$$

and hence

$$\nu(C, \Omega^2_X) = \frac{1}{9}\int_Y (1 - \zeta - \frac{\zeta^2}{2} c_1(C)) \cdot 2(\zeta + 2)c_1(C) = \frac{2}{3}\,\chi(C).$$

Finally, if $Y = S$ is a surface, then

$$\begin{aligned}\Omega^2_X|_S &= \Omega^2_S \oplus \Lambda^2 N^*_\zeta \oplus (\Omega^1_S \otimes N^*_\zeta)\\ &\cong \Omega^2_S \oplus \Lambda^2 TS \oplus (\Omega^1_S \otimes TS)\end{aligned}$$

and

$$\begin{aligned}\mathrm{ch}(\Omega^2_X|_S, f) = 1 - c_1(S) + \frac{1}{2}c_1^2(S) + \zeta^2(1 + c_1(S) + \frac{1}{2}c_1^2(S))\\ + \zeta(2 + c_1(S) + \frac{1}{2}c_1^2(S) - c_2(S))(2 - c_1(S) + \frac{1}{2}c_1^2(S) - c_2(S)).\end{aligned}$$

Multiplying this with (3.1) gives

$$\nu(S, \Omega^2_X) = \frac{1}{12}(3c_1^2(S) + 11c_2(S)).$$

The Lefschetz formula in this case is

$$2N + 2\chi(C) + \frac{1}{4}(3c_1^2(S) + 11c_2(S)) = 3\,L(f, \Omega^2_X).$$

□

Proposition 3.2.6. *Assume that X is of $K3^{[2]}$-type. Then $\operatorname{tr} f^*|H^{1,1}(X) = 3s$ for some integer $-3 \leq s \leq 7$, and*

$$\begin{aligned}2N + \chi(C) &= 3s(s+3)\\ \chi(C) + 2c_2(S) &= 6s^2\\ c_1^2(S) + c_2(S) &= 6s(s-1).\end{aligned}$$

Moreover, any $-3 \leq s \leq 7$ occurs for some automorphism.

Proof. Let

$$H^{1,1}(X) = H_1 \oplus H_\zeta \oplus H_{\zeta^2}$$

be the eigenspace decomposition with respect to f^*. Since f^* is defined over $\mathbb{Z}$, we have $H_{\zeta^2} = \overline{H}_\zeta$, and in particular $h_\zeta := \dim H_\zeta = \dim H_{\zeta^2}$. Since

$$h_1 := \dim H_1 = 21 - 2h_\zeta, \tag{3.4}$$

this shows that

$$\operatorname{tr} f^*|H^{1,1}(X) = h_1 + h_\zeta(\zeta + \zeta^2) = h_1 - h_\zeta = 3(7 - h_\zeta) =: 3s$$

for some number $s \leq 7$. From (3.4) we obtain $h_\zeta \leq 10$ and hence $s \geq -3$.

For a manifold of $K3^{[2]}$-type, the space $H^{p,q}(X)$ vanishes if $p+q$ is odd. Moreover, the product with $\overline{\omega}$ defines an isomorphism $H^{1,1}(X) \to H^{1,3}(X)$. Therefore, the holomorphic Lefschetz number is

$$L(f, \Omega^1_X) = -\operatorname{tr} f^*|H^{1,1}(X) - \operatorname{tr} f^*|H^{1,3}(X) = -3s - 3s \cdot \zeta^2 = 3s \cdot \zeta.$$

By [Ver96, Thm. 1.5], the cup product defines an isomorphism

$$\operatorname{Sym}^2 H^2(X,\mathbb{C}) \xrightarrow{\sim} H^4(X,\mathbb{C}).$$

Since this preserves the Hodge structure, we have

$$H^{2,2}(X) = (H^{2,0}(X) \otimes H^{0,2}(X)) \oplus \operatorname{Sym}^2 H^{1,1}(X).$$

The eigenspace decomposition $H^{2,2}(X) =: W_1 \oplus W_\zeta \oplus W_{\zeta^2}$ is given by

$$\begin{aligned} W_1 &= (H^{2,0}(X) \otimes H^{0,2}(X)) \oplus (H_\zeta \otimes H_{\zeta^2}) \oplus \operatorname{Sym}^2 H_1, \\ W_\zeta &= (H_1 \otimes H_\zeta) \oplus \operatorname{Sym}^2 H_{\zeta^2}, \\ W_{\zeta^2} &= (H_1 \otimes H_{\zeta^2}) \oplus \operatorname{Sym}^2 H_\zeta. \end{aligned}$$

This gives

$$\begin{aligned} \dim W_1 &= 1 + h_\zeta^2 + \frac{1}{2} h_1(h_1+1) \\ \dim W_\zeta &= h_1 h_\zeta + \frac{1}{2} h_\zeta(h_\zeta+1). \end{aligned}$$

Substituting $h_1 = 7+2s$ and $h_\zeta = 7-s$, we obtain

$$\operatorname{tr} f^*|H^{2,2}(X) = \dim W_1 - \dim W_\zeta = \frac{3}{2}s(3s+1) + 1.$$

Moreover, the product with $\overline{\omega}^2$ defines an isomorphism $H^{2,0}(X) \xrightarrow{\sim} H^{2,4}(X)$. Thus the Lefschetz number is given by

$$\begin{aligned} L(f,\Omega_X^2) &= \operatorname{tr} f^*|H^{2,0}(X) + \operatorname{tr} f^*|H^{2,2}(X) + \operatorname{tr} f^*|H^{2,4}(X) \\ &= \zeta + \frac{3}{2}s(3s+1) + 1 + \zeta(\zeta^2)^2 = \frac{3}{2}s(3s+1). \end{aligned}$$

Simplifying the equations from Proposition 3.2.5, we obtain the equations given in the statement.

The last claim is equivalent to the existence of an automorphism with invariant lattice of any odd rank $1 \le h_1 \le 21$. For any odd $3 \le h_1 \le 21$, there exists a non-symplectic automorphism of a K3 surface with invariant lattice of rank $h_1 - 1$ by [AS08, Prop. 3.2], hence these numbers are covered by natural automorphisms. For $h_1 = 1$, consider Example 3.1.7 (i): the fixed locus contains only the Fano surface S, which implies $N = \chi(C) = 0$ and hence $s = 0$ or $s = -3$. Since S is of general type, we have $c_1^2(S) > 0$ and therefore $s = -3$ and $h_1 = 7 + 2s = 1$. □

Remark 3.2.7. It is also possible to apply the topological Lefschetz formula [Pet86]

$$\chi(X^f) = \sum_{i=0}^{4} (-1)^i \operatorname{tr} f^*|H^i(X,\mathbb{R}),$$

where the action on $H^4(X,\mathbb{R})$ can be computed as above. This is done in [BCS14, 3.2] for automorphisms of arbitrary prime order. However, the four equations we then have are linearly dependent. In fact, the following example shows that another equation of this form cannot exist.

Example 3.2.8. Consider a K3 surface S with a non-symplectic automorphism $\sigma : S \to S$ of order 3, and let the numbers n, k, g be as in Theorem 3.1.4. Let $f = \sigma^{[2]} : X \to X$ be the natural automorphism, where $X = S^{[2]}$. For a curve C_g of genus g, the Chern numbers of the surface $\mathbb{P}^1 \times C_g$ are

$$c_1^2(\mathbb{P}^1 \times C_g) = 8(1-g),$$
$$c_2(\mathbb{P}^1 \times C_g) = 4(1-g).$$

By [Oxb00], those of the symmetric square of C_g are given by

$$c_1^2(C_g^{(2)}) = (9-4g)(1-g),$$
$$c_2(C_g^{(2)}) = (3-2g)(1-g).$$

We have

$$3s = \operatorname{tr} f^*|H^{1,1}(X) = \operatorname{tr} f^*|H^2(X,\mathbb{R}) + 1 = \operatorname{tr} \sigma^*|H^2(S,\mathbb{R}) + 2 = 3n - 6,$$

and hence $s = n - 2$. Together with $g = 3 + k - n$, we obtain

$$\begin{array}{lcl} N & = & (s+1)(s+2)/2 \\ \chi(C) & = & 2s^2 + 6s - 2 \\ c_1^2(S) & = & 4s^2 - 3s - 1 \\ c_2(S) & = & 2s^2 - 3s + 1. \end{array}$$

Using this, we can verify the computations. Now consider the case $s = 0$. For natural automorphisms the fixed locus has the topological invariants

$$(N, \chi(C), c_1^2(S), c_2(S)) = (1, -2, -1, 1).$$

In Example 3.1.7 (v), the fixed locus was given by three elliptic curves, so in this case we have $(N, \chi(C), c_1^2(S), c_2(S)) = (0, 0, 0, 0)$. By Proposition 3.2.6, this implies $s = 0$. This shows that there cannot exist another equation of this type.

However, we remark that in [BNWS13, Thm. 1.2], the authors obtain another formula for non-symplectic automorphisms of prime order $3 \leq p \leq 19$, $p \neq 5$, relating the number $h^*(X^f, \mathbb{F}_p)$ to the rank of the invariant lattice and the length of its discriminant group.

Chapter 4

Moduli spaces of non-symplectic involutions

In this chapter, we study moduli spaces of pairs (X, i), where X is a manifold of $K3^{[n]}$-type and $i : X \to X$ is a non-symplectic involution. We will see that, unlike for K3 surfaces, the invariant lattice does not necessarily determine the deformation type of the involution. In Section 4.8, we will give a purely lattice theoretic criterion for deformation equivalence. Another difference from the K3 case is the fact that Hausdorff moduli spaces do not always exist. In Section 4.9 we show that a quasi-projective (and in particular Hausdorff) moduli space does exist, if we disregard a certain class of involutions which corresponds to a codimension 1 analytic subvariety of the local deformation space.

4.1 Non-symplectic involutions

Recall that a non-symplectic involution of an irreducible symplectic manifold is a biholomorphic automorphism $i : X \to X$ with $i \circ i = \mathrm{id}_X$ and $i^*\omega = -\omega$.

Definition 4.1.1. A *family* $(\pi, I) : \mathcal{X} \to S$ of non-symplectic involutions over a connected smooth analytic space S consists of

(i) a smooth and proper family $\pi : \mathcal{X} \to S$ of irreducible symplectic manifolds, and

(ii) a holomorphic involution $I : \mathcal{X} \to \mathcal{X}$ with $\pi \circ I = \pi$, such that for every $t \in S$, the induced involution $I_t : X_t \to X_t$ is non-symplectic.

Definition 4.1.2. Let $i_1 : X_1 \to X_1$ and $i_2 : X_2 \to X_2$ be non-symplectic involutions.

(i) The pairs (X_1, i_1) and (X_2, i_2) are *isomorphic*, if there exists an isomorphism $f : X_1 \to X_2$ with $i_2 \circ f = f \circ i_1$.

(ii) The pairs (X_1, i_1) and (X_2, i_2) are *deformation equivalent*, if there exists a family $(\pi, I) : \mathcal{X} \to S$ of non-symplectic involutions and points $t_j \in S$ with $(X_{t_j}, I_{t_j}) \cong (X_j, i_j)$ for $j = 1, 2$.

Let $i : X \to X$ be a non-symplectic involution. The induced map

$$i^* : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is a monodromy operator and an isometry with respect to the Beauville–Bogomolov form. By the following theorem, the involution i is determined by i^* for manifolds of $K3^{[n]}$-type.

Theorem 4.1.3. *Let X be an irreducible symplectic manifold of $K3^{[n]}$-type and $f : X \to X$ an automorphism acting trivially on $H^2(X, \mathbb{Z})$. Then $f = \mathrm{id}_X$.*

Proof. For the special case $X = S^{[n]}$ for some K3 surface S, this was shown by Beauville [Bea83a, Prop. 10]. The general case follows from [KV98, Cor. 6.9] (see also [Mar10, Section 1.2]). □

The most important invariant of the pair (X, i) is the invariant sublattice

$$H^2(X, \mathbb{Z})^i = \{h \in H^2(X, \mathbb{Z}) : i^*(h) = h\} \subset H^2(X, \mathbb{Z}).$$

Let ω be the symplectic form of X. For every invariant class $h \in H^2(X, \mathbb{Z})^i$, we have

$$(\omega, h) = (i^*(\omega), i^*(h)) = -(\omega, h)$$

and hence

$$(\omega, H^2(X, \mathbb{Z})^i) = 0. \tag{4.1}$$

This shows that $H^2(X, \mathbb{Z})^i \subset H^{1,1}(X, \mathbb{Z})$. If $x \in H^2(X, \mathbb{R})$ is a Kähler class, then $i^*(x)$ is a Kähler class and therefore

$$\tilde{x} := x + i^*(x) \in H^2(X, \mathbb{R})^i$$

is an invariant Kähler class. Since $(\tilde{x}, \tilde{x}) > 0$, this implies that

$$H^2(X, \mathbb{R})^i = H^2(X, \mathbb{Z})^i \otimes \mathbb{R} \subset H^{1,1}(X, \mathbb{R})$$

is hyperbolic, and therefore also the invariant lattice $H^2(X, \mathbb{Z})^i$. By Huybrechts' projectivity criterion, this shows that any irreducible symplectic manifold admitting a non-symplectic involution is projective.

4.2 Period map

From now on, we will only consider manifolds of $K3^{[n]}$-type for some fixed number n and write

$$L := L_n = 3U \oplus 2E_8 \oplus \langle 2-2n \rangle$$

for the $K3^{[n]}$ lattice.

Definition 4.2.1. A sublattice $M \subset L$ is called *admissible*, if

(i) M is hyperbolic,

(ii) there exists an involution $\iota_M \in \mathrm{Mon}(L)$ such that $M = L^{\iota_M}$.

Note that property (ii) implies that $M \subset L$ is a primitive sublattice. Moreover, we emphasize that we always consider M as a sublattice of L. In general, the abstract lattice M admits non-isometric embeddings into L (for examples, we refer to [BCS14, Example. 8.6] or Chapter 5).

If $i : X \to X$ is a non-symplectic involution, then any marking

$$\alpha : H^2(X, \mathbb{Z}) \to L$$

maps the invariant sublattice $H^2(X,\mathbb{Z})^i \subset H^2(X,\mathbb{Z})$ to some admissible sublattice $M \subset L$. This follows from the preceding section and the fact that the monodromy group $\mathrm{Mon}^2(X) \subset O(H^2(X,\mathbb{Z}))$ is a normal subgroup.

In the case $n = 2$, admissible sublattices have been classified by Boissière–Camere–Sarti in [BCS14]. Moreover, it is shown that every such lattice is isometric to the invariant sublattice $H^2(X,\mathbb{Z})^i \subset H^2(X,\mathbb{Z})$ for some non-symplectic involution $i : X \to X$ of a $K3^{[2]}$-type manifold [BCS14, Prop. 8.2]. We will see that the same is true for $n > 2$.

We now fix a connected component $\mathfrak{M}_L^0$ of the moduli space of marked pairs and denote by $P_0 : \mathfrak{M}_L^0 \to \Omega_L$ the restriction of the period map.

Definition 4.2.2. Let $M \subset L$ be an admissible sublattice with corresponding involution $\iota_M \in \mathrm{Mon}(L)$ and let $i : X \to X$ be a non-symplectic involution.

(i) A marking $\alpha : H^2(X,\mathbb{Z}) \to L$ is called *admissible* (for M), if $\alpha \circ i^* = \iota_M \circ \alpha$ and $(X, \alpha) \in \mathfrak{M}_L^0$.

(ii) The pair (X, i) is called *of type M*, if there exists a marking $\alpha : H^2(X,\mathbb{Z}) \to L$ which is admissible for M.

If $(\pi, I) : \mathcal{X} \to S$ is a family of non-symplectic involutions, then for $t \in S$ the map

$$I_t^* : H^2(X_t, \mathbb{Z}) \to H^2(X_t, \mathbb{Z})$$

is given by restriction of the morphism $R^2\pi_*\mathbb{Z} \to R^2\pi_*\mathbb{Z}$ of local systems induced by I. Thus, if (X_1, i_1) and (X_2, i_2) are deformation equivalent, then there exists a parallel transport operator

$$g : H^2(X_1, \mathbb{Z}) \to H^2(X_2, \mathbb{Z}), \qquad g \circ i_1^* = i_2^* \circ g. \tag{4.2}$$

Therefore, if α is an admissible marking of (X_2, i_2), then $\alpha \circ g$ is an admissible marking of (X_1, i_1). In particular, any deformation of a pair of type M is again of type M.

Remark 4.2.3. The definition of pairs of type M depends on the choice of $\mathfrak{M}_L^0$ in the following way: The group $O(L)/\operatorname{Mon}(L)$ acts simply transitively on set the of connected components of $\mathfrak{M}_L$ (see [Mar11, Rem. 7.6]). Hence for any non-symplectic involution $i : X \to X$ with invariant lattice isometric to $M \subset L$, the pair (X, i) is of type M' for some sublattice $M' \subset L$ isometric to $M \subset L$. Moreover, if (X, i) is of type M, and $M' \subset L$ is another admissible sublattice, then (X, i) is of type M' if and only if there exists an isometry $\sigma \in \operatorname{Mon}(L)$ such that $\sigma(M) = M'$.

Let us call two pairs (X_1, i_1) and (X_2, i_2) of the same *lattice type*, if there exists a parallel transport operator as in (4.2). As noted before, being of the same lattice type is invariant under deformation. Once a component $\mathfrak{M}_L^0$ is chosen, we can identify lattice types and $\operatorname{Mon}(L)$-orbits of admissible sublattices $M \subset L$.

Note that if $n = 2$ or $n - 1$ is a prime power, then $O(L)$-orbits and $\operatorname{Mon}(L)$-orbits of sublattices coincide, since $\operatorname{Mon}(L) \to O(L)/\{+1, -1\}$ is surjective. We will see in Chapter 5 that this is still true for most lattices, even if the monodromy group is smaller. In these cases, the definition of pairs of type M does not depend on the choice of $\mathfrak{M}_L^0$.

Let

$$\mathcal{M}_M := \{(X, i) : (X, i) \text{ is a pair of type } M\}/\cong .$$

For now, we will consider $\mathcal{M}_M$ only as a set.

Let (X, i) be a pair of type M and $\alpha : H^2(X, \mathbb{Z}) \to L$ be an admissible marking. Using (4.1), we obtain

$$P_0(X, \alpha) \in \Omega_{M^\perp} \subset \Omega_L.$$

Consider the subgroup

$$\begin{aligned}\Gamma(M) &:= \{\sigma \in \operatorname{Mon}(L) : \sigma \circ \iota_M = \iota_M \circ \sigma\} \\ &= \{\sigma \in \operatorname{Mon}(L) : \sigma(M) = M\}.\end{aligned}$$

If (X, i) and (Y, j) are of type M with admissible markings $\alpha : H^2(X, \mathbb{Z}) \to L$ and $\beta : H^2(Y, \mathbb{Z}) \to L$, and $f : (X, i) \to (Y, j)$ is an isomorphism, then f^* is a Hodge isometry, and therefore

$$P_0(X, \alpha) = \sigma(P_0(Y, \beta)), \quad \text{where} \quad \sigma := \alpha \circ f^* \circ \beta^{-1}.$$

Since f^* is a parallel transport operator and (X, α) and (Y, β) belong to the same connected component of $\mathfrak{M}_L$, we have $\sigma \in \mathrm{Mon}(L)$. Furthermore, using $i^* \circ f^* = f^* \circ j^*$, we obtain

$$\begin{aligned} \iota_M \circ \sigma &= \iota_M \circ \alpha \circ f^* \circ \beta^{-1} = \alpha \circ i^* \circ f^* \circ \beta^{-1} \\ &= \alpha \circ f^* \circ j^* \circ \beta^{-1} = \alpha \circ f^* \circ \beta^{-1} \circ \iota_M = \sigma \circ \iota_M \end{aligned}$$

and hence $\sigma \in \Gamma(M)$. Thus the period map induces a map

$$P_M : \mathcal{M}_M \quad \longrightarrow \quad \Omega_{M^\perp}/\Gamma_{M^\perp}, \tag{4.3}$$

where $\Gamma_{M^\perp} \subset O(M^\perp)$ is the image of the restriction homomorphism

$$\Gamma(M) \to O(M^\perp).$$

Proposition 4.2.4. *$\Gamma_{M^\perp} \subset O(M^\perp)$ is a finite index subgroup.*

Proof. By Lemma 2.4.2, any isometry inside the finite index subgroup $\widetilde{O}(M^\perp) \subset O(M^\perp)$ extends to an isometry $\sigma \in \widetilde{O}(L)$. By Lemma 1.4.7, either σ or $-\sigma$ belongs to $\mathrm{Mon}(L)$ and hence to $\Gamma(M)$. □

Since $\mathrm{sign}(M^\perp) = (2, r(M^\perp)-2)$, the finite index subgroup $\Gamma_{M^\perp} \subset O(M^\perp)$ acts properly discontinuously on $\Omega_{M^\perp}$, and the quotient $\Omega_{M^\perp}/\Gamma_{M^\perp}$ is a quasi-projective variety by [BB66, Thm. 10.4 and Thm 10.11].

Assume that $(\pi, I) : \mathcal{X} \to S$ is a holomorphic family of involutions of type M. Then the holomorphicity of the ordinary period map implies that the induced map

$$\begin{aligned} S &\to \Omega_{M^\perp}/\Gamma_{M^\perp} \\ t &\mapsto P_M(X_t, I_t) \end{aligned}$$

is holomorphic.

4.3 Deformation theory of involutions

The local deformation theory of non-symplectic involutions has been described by Beauville [Bea11, Thm. 2]. A more detailed discussion for automorphisms of arbitrary finite order on irreducible symplectic manifolds is given in [BCS14, Section 4]. We briefly recall the facts.

Let (X, i) be a pair of type M and $\pi : \mathcal{X} \to \mathrm{Def}(X)$ be the Kuranishi family of $X = \pi^{-1}(0)$. The involution of i on X extends holomorphically to an involution $I : \mathcal{X} \to \mathcal{X}$, and since the Kuranishi family is universal, this defines an action of i on $\mathrm{Def}(X)$. The deformation space $\mathrm{Def}(X)$ can be locally identified with $H^1(X, TX)$ and the actions of i on these spaces concide under this identification. Therefore, the invariant subspace $\mathrm{Def}(X, i) := \mathrm{Def}(X)^i$ is smooth. Moreover, the symplectic form defines an isomorphism $TX \to \Omega_X^1$, which maps the invariant subspace of

$H^1(X, TX)$ to the (-1)-eigenspace of $H^1(X, \Omega_X^1)$. After choosing an admissible marking, the period map $\mathrm{Def}(X) \hookrightarrow \Omega_L$ thus restricts to an open embedding

$$\mathrm{Def}(X, i) \hookrightarrow \Omega_{M^\perp}.$$

In particular, the dimension of $\mathrm{Def}(X, i)$ is $21 - r(M)$. Moreover, the Kuranishi family restricts to a family

$$\pi' : \mathcal{X}' \to \mathrm{Def}(X, i),$$

such that $I' := I|_{\mathcal{X}'}$ preserves the fibres of π'. The family $(\pi', I') \to \mathrm{Def}(X, i)$ is a universal deformation of (X, i).

Example 4.3.1. Let $i : S \to S$ be a non-symplectic involution and

$$i^{[n]} : S^{[n]} \to S^{[n]}$$

be the natural involution. Any deformation of (S, i) induces a deformation of $(S^{[n]}, i^{[n]})$. On the other hand, we have

$$H^{1,1}(S^{[2]})^{i^{[2]}} = j(H^{1,1}(S)^i) \oplus \mathbb{C}e,$$

hence j maps $\mathrm{Def}(S, i)$ onto $\mathrm{Def}(S^{[2]}, i^{[2]})$. Every small deformation of $(S^{[2]}, i^{[2]})$ is induced by S. Note that this is different from the symplectic case [Cam12, Prop. 7].

4.4 Results for K3 surfaces

For K3 surfaces, one has the following results due to Nikulin and Yoshikawa.

Theorem 4.4.1. *(i) The isometry class of the invariant lattice $M \subset L_{K3}$ determines the deformation type of a non-symplectic involution.*

(ii) The period map $P_M : \mathcal{M}_M \to \Omega_{M^\perp}/\Gamma_{M^\perp}$ is injective and its image is a Zariski-open subset $\Omega^0_{M^\perp}/\Gamma_{M^\perp}$. In particular, $\Omega^0_{M^\perp}/\Gamma_{M^\perp}$ is a coarse moduli space of pairs of type M.

Proof. (i) was shown by Nikulin [Nik80b, Rem. 4.5.3] and (ii) by Yoshikawa [Yos04, Thm. 1.8]. □

Remark 4.4.2. In fact, Yoshikawa does not impose the condition $(X, \alpha) \in \mathfrak{M}^0_L$ for admissible markings, and considers the quotient of $\Omega_{M^\perp}$ by the image of

$$\{\sigma \in O(L_{K3}) : \sigma \circ \iota_M = \iota_M \circ \sigma\} \to O(M^\perp).$$

For K3 surfaces, this is equivalent to the period map given above, since

$$\mathrm{Mon}(L_{K3}) \to O(L_{K3})/\{+1, -1\}$$

is surjective, and $-\operatorname{id}_{M^\perp}$ acts trivially on $\Omega_{M^\perp}$.

We further remark that in this case there is an isomorphism

$$\Omega_{M^\perp}/\Gamma_{M^\perp} \cong \Omega^+_{M^\perp}/\Gamma^+_{M^\perp},$$

where $\Gamma^+_{M^\perp} \subset \Gamma_{M^\perp}$ is the subgroup that preserves the connected components $\Omega^+_{M^\perp}$ and $\Omega^-_{M^\perp}$ of $\Omega_{M^\perp}$. Indeed, this follows from the fact, that there exists an isometry $\xi \in \Gamma_{M^\perp}$ with $\xi(\Omega^+_{M^\perp}) = \Omega^-_{M^\perp}$, see [Nik83, Rem. 4.5.3].

We will see that both statements of Theorem 4.4.1 are no longer true for $K3^{[n]}$-type manifolds. There can be more than one deformation type of pairs of type M, and even when the period map P_M is restricted to involutions of a fixed deformation type, it need not be generically injective.

In Section 4.8, we will give a description of the deformation types in terms of a chamber decomposition of the positive cone of M. Once we restrict to involutions corresponding to a given deformation type $\mathcal{K}$, we will be able to use a finer period map

$$P_{M,\mathcal{K}} : \mathcal{M}_{M,\mathcal{K}} \to \Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}}$$

for some finite index subgroup $\Gamma_{M^\perp,\mathcal{K}} \subset \Gamma^+_{M^\perp}$.

4.5 Kähler cone

In this section, we present the description of the Kähler-type chambers given by Amerik–Verbitsky [AV14b]. A similar result for the Kähler cone was shown by Mongardi [Mon13, Thm. 1.3].

Definition 4.5.1. A rational homology class $z \in H^{1,1}(X, \mathbb{Q})$ with $(z, z) < 0$ is called *monodromy birationally minimal*, if there exists a birational map $f : X \dashrightarrow \widetilde{X}$ and a monodromy operator $g \in \operatorname{Mon}^2(X)$, such that the hyperplane $g(z)^\perp$ contains a face of $f^*\mathcal{K}_{\widetilde{X}}$.

Theorem 4.5.2 (Amerik–Verbitsky). *Let $z \in H^{1,1}(X, \mathbb{Z})$ be a monodromy birationally minimal class on X, and (X', z') a deformation of (X, z), such that z' is of type $(1, 1)$. Then z' is monodromy birationally minimal.*

Proof. [AV14a, Thm. 2.16] □

Theorem 4.5.3 (Amerik–Verbitsky). *The Kähler-type chambers of X are the connected components of*

$$\mathcal{C}_X \setminus \bigcup_z z^\perp,$$

where the union is taken over all monodromy birationally minimal classes on X.

Proof. [AV14b, Thm. 6.2] □

Let $\Delta(X) \subset H^{1,1}(X,\mathbb{Z})$ be the subset of all primitive integral classes which are monodromy birationally minimal. We call such classes *wall divisors* (as in [Mon13]). Note that some rational multiple of any monodromy birationally minimal class is a wall divisor.

Proposition 4.5.4. *For any connected component $\mathfrak{M}_L^0$ of the moduli space of marked pairs of $K3^{[n]}$-type, there exists a subset $\Delta(L) \subset L$ with the following properties:*

(i) For any $(X,\alpha) \in \mathfrak{M}_L^0$, we have

$$\Delta(X) = \alpha^{-1}(\Delta(L)) \cap H^{1,1}(X,\mathbb{Z}).$$

(ii) The group $\mathrm{Mon}(L)$ *acts on* $\Delta(L)$ *with a finite number of orbits.*

Proof. Let

$$\Delta(L) := \{\alpha(D) : (X,\alpha) \in \mathfrak{M}_L^0 \text{ and } D \in \Delta(X)\} \subset L.$$

Let $D = \alpha^{-1}(\alpha'(D'))$, where $(X',\alpha') \in \mathfrak{M}_L^0$ and $D' \in \Delta(X')$. Then $\alpha^{-1} \circ \alpha'$ is a parallel transport operator, and hence (X,D) is a deformation of (X',D'). If $D \in H^{1,1}(X,\mathbb{Z})$, then $D \in \Delta(X)$ by Theorem 4.5.2. The other inclusion follows from the definition of $\Delta(L)$. This shows (i).

The group $\mathrm{Mon}(L)$ clearly acts on $\Delta(L)$. The finiteness of orbits follows from [BHT13, Prop. 2], as explained in [AV14b, §6.2]: There exists a constant $C_n > 0$, such that for every manifold X of $K3^{[n]}$-type, every wall divisor $D \in \Delta(X)$ satisfies $|(D,D)| < C_n$. Therefore, we have $|(\delta,\delta)| < C_n$ for any $\delta \in \Delta(L)$. Since the monodromy group $\mathrm{Mon}(L) \subset O(L)$ is a finite index subgroup, the claim follows from Lemma 2.4.1. ☐

The set $\Delta(L)$ has been explicity determined for $n = 2,3,4$ by Mongardi [Mon13]. We will only need the explicit description for $n = 2$:

Proposition 4.5.5. *For X of $K3^{[2]}$-type, the wall divisors are given by*

$$\Delta(X) = \{D \in H^{1,1}(X,\mathbb{Z}) : (D,D) = -2 \text{ or } (D,D) = -10,\ \mathrm{div}_{H^2(X,\mathbb{Z})}(D) = 2\}.$$

Proof. Hassett–Tschinkel [HT09, Thm. 23] showed that every wall divisor is of this form, and Markman [Mar13, Thm. 1.11] and Mongardi [Mon13, Prop. 2.12] showed that every such divisor is a wall divisor. ☐

In particular, in this case the set $\Delta(L)$ does not depend on $\mathfrak{M}_L^0$, and we have

$$\Delta(L) = \{\delta \in L : (\delta,\delta) = -2, \text{ or } (\delta,\delta) = -10,\ \mathrm{div}(\delta) = 2\}.$$

4.6 Stable invariant Kähler cone

For a non-symplectic involution $i : X \to X$ let

$$\mathcal{C}_X^i := \{x \in \mathcal{C}_X : i^*(x) = x\}$$

be the invariant positive cone and

$$\Delta^i(X) := \{D \in \Delta(X) : i^*(D) = D\}$$

the set of invariant wall divisors of (X, i). It follows from Theorem 4.5.3 that the invariant Kähler cone $\mathcal{K}_X^i = \mathcal{K}_X \cap \mathcal{C}_X^i$ of (X, i) is contained in a connected component of

$$\mathcal{C}_X^i \setminus \bigcup_{D \in \Delta^i(X)} D^\perp. \tag{4.4}$$

Definition 4.6.1. The *stable invariant Kähler cone* $\widetilde{\mathcal{K}}_X^i$ of (X, i) is the component of (4.4) containing the invariant Kähler cone of (X, i).

We will give a geometric interpretation of $\widetilde{\mathcal{K}}_X^i$ in Proposition 4.8.4, and show in Proposition 4.8.3, that the stable invariant Kähler cone, unlike the invariant Kähler cone, is stable under deformation.

The reason the period map P_M need not be injective, is that a pair (X, i) can have a different birational model $f : X \dashrightarrow \widetilde{X}$ such that the induced birational involution

$$\tilde{i} := f \circ i \circ f^{-1} : \widetilde{X} \to \widetilde{X}$$

is again biregular. In this case we have $P_M(X, i) = P_M(\widetilde{X}, \tilde{i})$, and in Section 4.10 we will see that, at least for $n = 2$, also the converse is true.

If $\mathcal{K}_X$ and $f^*\mathcal{K}_{\widetilde{X}}$ are separated by a wall $D^\perp$ for an invariant wall divisor $D \in \Delta^i(X)$, then D remains of type $(1, 1)$ for any deformation of the pair (X, i). In this case the two birational models deform into different families. (At least locally; globally the families can be the same, as we will see in Example 4.9.7.)

If D belongs to $\Delta(X) \setminus \Delta^i(X)$, then the corresponding wall vanishes under some deformation of (X, i), and the two pairs (X, i) and $(\widetilde{X}, \tilde{i})$ deform into the same family. An example will be given at the end of this section. In this case, the pairs (X, i) and $(\widetilde{X}, \tilde{i})$ are inseparable in the following sense.

Definition 4.6.2. Two non-isomorphic pairs (X, i) and $(\widetilde{X}, \tilde{i})$ are called *inseparable*, if their universal deformations $(\mathcal{X}, I) \to \mathrm{Def}(X, i)$ and $(\widetilde{\mathcal{X}}, \widetilde{I}) \to \mathrm{Def}(\widetilde{X}, \tilde{i})$ (considered as germs) contain isomorphic fibers.

The existence of inseparable pairs clearly implies that a Hausdorff moduli space cannot exist. Our goal is to show, that we can obtain a quasi-projective (and in particular Hausdorff) moduli space, if we remove the hyperplanes $\delta^\perp \subset \Omega_{M^\perp}$ for certain $\delta \in \Delta(L) \setminus \Delta(M)$. While the set of all such hyperplanes is not locally finite in the period domain, we will see that the induced involution $\tilde{\imath}$ can only be biregular, if the hyperplane $D^\perp \subset \mathcal{C}_X$ meets the invariant positive cone. This, together with Lemma 4.6.5, motivates the following definition.

Definition 4.6.3. We denote by $L_M \subset L$ the set of elements $\delta \in L$ such that

$$\operatorname{sign}(M \cap \delta^\perp) = (1, r(M) - 2),$$
$$\operatorname{sign}(M^\perp \cap \delta^\perp) = (2, r(M^\perp) - 3).$$

Definition 4.6.4. The *positive cone* of M is

$$\widetilde{\mathcal{C}}_M := \{x \in M_\mathbb{R} : (x, x) > 0\}.$$

Lemma 4.6.5. *For $\delta \in L$ the following properties are equivalent:*

(i) $\delta \in L_M$

(ii) δ *satisfies the following conditions:*

- *(a)* $\delta \notin M$,
- *(b)* $\delta \notin M^\perp$,
- *(c)* $\Omega_{M^\perp} \cap \delta^\perp \neq \emptyset$,
- *(d)* $\widetilde{\mathcal{C}}_M \cap \delta^\perp \neq \emptyset$.

(iii) Let $\delta_M \in M_\mathbb{Q}$ and $\delta_{M^\perp} \in M^\perp_\mathbb{Q}$ such that $\delta = \delta_M + \delta_{M^\perp}$. Then

$$(\delta_M, \delta_M) < 0, \qquad (\delta_{M^\perp}, \delta_{M^\perp}) < 0.$$

Proof. First assume that $\delta \in L_M$. Then (a) and (b) follow from $M \cap \delta^\perp \neq 0$ and $M^\perp \cap \delta^\perp \neq 0$. Since $M \cap \delta^\perp$ is hyperbolic, we have

$$\widetilde{\mathcal{C}}_M \cap \delta^\perp = \widetilde{\mathcal{C}}_{M \cap \delta^\perp} \neq \emptyset,$$

and since $M^\perp \cap \delta$ has two positive squares, we have

$$\Omega_{M^\perp} \cap \delta^\perp = \Omega_{M^\perp \cap \delta^\perp} \neq \emptyset.$$

Conversely, assume that $\delta \in L$ satisfies (a)–(d). The sublattice $\delta^\perp \cap M \subset M$ is hyperbolic, parabolic or negative definite. The latter two cases are excluded by condition (d). Since $\delta \notin M^\perp$, this shows that $\operatorname{sign}(M) = (1, r(M) - 2)$. Condition (c) implies that $M^\perp \cap \delta^\perp$ has two positive squares and together with (a) we obtain $\delta \in L_M$.

We now show the equivalence of (i) and (iii). If δ satisfies (iii), then $\delta \in L_M$ is a consequence of the orthogonal decompositions

$$\begin{aligned} M_{\mathbb{Q}} &= (M_{\mathbb{Q}} \cap \delta^{\perp}) \oplus \mathbb{Q}\,\delta_M, \\ M_{\mathbb{Q}}^{\perp} &= (M_{\mathbb{Q}}^{\perp} \cap \delta^{\perp}) \oplus \mathbb{Q}\,\delta_M^{\perp}. \end{aligned} \tag{4.5}$$

Conversely, if $\delta \in L_M$, then since $M \cap \delta^{\perp}$ is non-degenerate and

$$r(M \cap \delta^{\perp}) = r(M) - 1,$$

we have $(\delta_M, \delta_M) \neq 0$. Therefore decompositions (4.5) hold and hence we have $(\delta_M, \delta_M) < 0$ and $(\delta_{M^{\perp}}, \delta_{M^{\perp}}) < 0$. □

For any sublattice $N \subset L$ let

$$\Delta(N) := \Delta(L) \cap N.$$

Moreover, let

$$\Delta_M(L) := L_M \cap \Delta(L).$$

Lemma 4.6.6. *The collections of hyperplanes*

$$\{\delta^{\perp} \subset \Omega_{M^{\perp}} : \delta \in \Delta_M(L)\}$$

and

$$\{\delta^{\perp} \subset \Omega_{M^{\perp}} : \delta \in \Delta(M^{\perp})\}$$

are locally finite in $\Omega_{M^{\perp}}$.

Proof. Since $\mathrm{Mon}(L)$ acts on $\Delta(L)$, the group $\Gamma_{M^{\perp}}$ acts on $\Delta(M^{\perp})$. There are only finitely many possible values (δ, δ) for $\delta \in \Delta(M^{\perp}) \subset \Delta(L)$ by Proposition 4.5.4 (ii), and since $\Gamma_{M^{\perp}} \subset O(M^{\perp})$ is a finite index subgroup, Lemma 2.4.1 implies that $\Delta(M^{\perp})$ consists of finitely many $\Gamma_{M^{\perp}}$-orbits. The group $\Gamma_{M^{\perp}}$ acts properly discontinuously on $\Omega_{M^{\perp}}$, which means that the map

$$\begin{aligned} \Omega_{M^{\perp}} \times \Gamma_{M^{\perp}} &\to \Omega_{M^{\perp}} \times \Omega_{M^{\perp}} \\ (\eta, \sigma) &\mapsto (\eta, \sigma(\eta)) \end{aligned}$$

is proper. In particular, every orbit $\Gamma_{M^{\perp}} \cdot \delta^{\perp} \subset \Omega_{M^{\perp}}$ is closed and hence a locally finite union of hyperplanes. This shows the first claim.

Now let $\delta \in \Delta_M(L)$. Since M is the invariant lattice of an involution, the quotient $L/(M \oplus M^{\perp})$ is a 2-torsion group. Therefore, we have $2\delta = \delta_M + \delta_{M^{\perp}}$, where $\delta_M \in M$ and $\delta_{M^{\perp}} \in M^{\perp}$. By Lemma 4.6.5, we have $(\delta_M, \delta_M) < 0$ and $(\delta_{M^{\perp}}, \delta_{M^{\perp}}) < 0$. Again, there is only a finite number of possible values for $(\delta, \delta) < 0$, and since

$$4(\delta, \delta) = (\delta_M, \delta_M) + (\delta_{M^{\perp}}, \delta_{M^{\perp}})$$

the same is true for $(\delta_{M^{\perp}}, \delta_{M^{\perp}})$. The group $\Gamma_{M^{\perp}}$ acts on the set of such $\delta_{M^{\perp}}$, and since $\delta^{\perp} = \delta_{M^{\perp}}^{\perp} \subset \Omega_{M^{\perp}}$, the same argument as above applies. □

By the preceding lemma, the subsets

$$\widetilde{\mathcal{D}}_M := \bigcup_{\delta \in \Delta(M^\perp)} \delta^\perp \subset \Omega_{M^\perp}$$

and

$$\widetilde{\mathcal{D}}'_M := \bigcup_{\delta \in \Delta_M(L)} \delta^\perp \subset \Omega_{M^\perp}$$

are closed in $\Omega_{M^\perp}$. They are invariant under $\Gamma_{M^\perp}$ and their quotients

$$\mathcal{D}_M := \widetilde{\mathcal{D}}_M / \Gamma_{M^\perp} \quad \text{and} \quad \mathcal{D}'_M := \widetilde{\mathcal{D}}'_M / \Gamma_{M^\perp}$$

are Zariski-closed subsets of $\Omega_{M^\perp}/\Gamma_{M^\perp}$. The significance of these divisors is explained by the following Proposition.

Proposition 4.6.7. *Let (X, i) be a pair of type M.*

(i) $P_M(X, i) \notin \mathcal{D}_M$.

(ii) If $P_M(X, i) \notin \mathcal{D}'_M$, then we have $\widetilde{\mathcal{K}}^i_X = \mathcal{K}^i_X$.

Proof. Let $\alpha : H^2(X, \mathbb{Z}) \to L$ be an admissible marking.

Assume first that $P_M(X, \alpha) \in \delta^\perp$, where $\delta \in \Delta(M^\perp)$. Then $D := \alpha^{-1}(\delta)$ is a wall divisor on X which is orthogonal to the invariant lattice. This is impossible, since there exists an invariant ample class on X.

Now assume that $\mathcal{K}^i_X$ is strictly smaller than $\widetilde{\mathcal{K}}^i_X$. This implies that there exists an element $D \in \Delta(X) \setminus \Delta^i(X)$ such that $D^\perp$ has non-empty intersection with $\widetilde{\mathcal{K}}^i_X \subset \mathcal{C}^i_X$. In particular, the element $\delta := \alpha(D)$ satisfies $\delta^\perp \cap \widetilde{\mathcal{C}}_M \neq \emptyset$. Furthermore we have $\delta \notin M^\perp$ by part (i) and $\delta \notin M$ by the assumption $D \notin \Delta^i(X)$. Finally, $P(X, \alpha) \in \Omega_{M^\perp} \cap \delta^\perp$ shows that this intersection is non-empty, and we can apply Lemma 4.6.5 to obtain $\delta \in L_M$. Since $D \in \Delta(X)$, we have $\delta \in \Delta_M(L)$. □

We remark, that the converse of part (ii) is not true in general. In fact, as seen in the proof, the property $P_M(X, i) \in \mathcal{D}'_M$ only implies the existence of a wall divisor in $\Delta(X) \setminus \Delta(X)^i$, meeting the invariant positive cone, rather then the stable invariant Kähler cone. Once we have discussed the problem of deformation equivalence, we will define a more suitable divisor $\mathcal{D}_{\mathcal{K}}$, which allows us to give a necessary and sufficient condition for $\widetilde{\mathcal{K}}^i_X = \mathcal{K}^i_X$ in terms of the period map.

We now give an example where $\mathcal{K}^i_X$ is strictly smaller than $\widetilde{\mathcal{K}}^i_X$, and the chambers of $\widetilde{\mathcal{K}}^i_X$ correspond to birational models deforming into the same family.

Example 4.6.8. Let $n = 2$ and e_1, e_2 and f_1, f_2 be standard bases for the first two hyperbolic planes of $L = 3U \oplus 2E_8 \oplus \mathbb{Z}e$, where $(e, e) = -2$. We consider the involution acting by $\iota_M(e_i) = f_i$ on $2U$ and as $-\operatorname{id}$ on $U \oplus 2E_8 \oplus \mathbb{Z}e$. Then the invariant and coinvariant lattice are given by

$$\begin{aligned} M &= \mathbb{Z}(e_1 + f_1) + \mathbb{Z}(e_2 + f_2) \cong U(2) \\ M^\perp &= (\mathbb{Z}(e_1 - f_1) + \mathbb{Z}(e_2 - f_2)) \oplus U \oplus 2E_8 \oplus \mathbb{Z}e \\ &\cong U(2) \oplus U \oplus 2E_8 \oplus \mathbb{Z}e. \end{aligned}$$

Let $\delta := 2e_1 - 2e_2 + e$. We have $(\delta, \delta) = -10$, $\operatorname{div}(\delta) = 2$ and hence $\delta \in \Delta(L)$. Moreover, we can write $\delta = \delta_M + \delta_{M^\perp}$ with

$$\begin{aligned} \delta_M \quad &= e_1 + f_1 - e_2 - f_2 \quad && \in M \\ \delta_{M^\perp} \quad &= e_1 - f_1 - e_2 + f_2 + e \quad && \in M^\perp. \end{aligned}$$

Since $(\delta_M, \delta_M) = -4 < 0$ and $(\delta_{M^\perp}, \delta_{M^\perp}) = -6 < 0$, we have $\delta \in \Delta_M(L)$. By Lemma 4.6.5, the intersection $\Omega_{M^\perp} \cap \delta^\perp$ is non-empty and for a generic period point $\eta \in \delta^\perp \subset \Omega_{M^\perp}$, we have

$$L^{1,1}(\eta) = L \cap \eta^\perp = M + \mathbb{Z}\delta = M \oplus \mathbb{Z}\delta_{M^\perp} \cong U(2) \oplus \langle -6 \rangle$$

and in particular $\eta \notin \mathcal{D}_M$. Let $\mathcal{C}_\eta$ be the positive cone of $L^{1,1}(\eta, \mathbb{R})$, and $\mathcal{C}_1, \mathcal{C}_2$ be the chambers of the invariant positive cone $\mathcal{C}_\eta \cap M_\mathbb{R}$ which are separated by $\delta^\perp$.

Now $\eta \notin \mathcal{D}_M$ implies that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect some Kähler-type chambers of $\mathcal{C}_\eta$ (a precise argument will be given in the proof of Lemma 4.8.5).

By Theorem 1.4.14, there exist two marked manifolds $(X_1, \alpha_1), (X_2, \alpha_2) \in \mathfrak{M}_L^0$ with $P(X_j, \alpha_i) = \eta$ and

$$\emptyset \neq \alpha_j(\mathcal{K}_{X_j}) \cap M_\mathbb{R} \subset \mathcal{C}_j.$$

(In fact, one can see that equality holds, but we do not need this.) Since $\eta \in \Omega_{M^\perp}$,

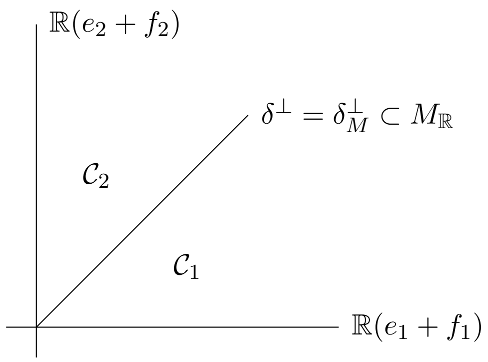


Figure 4.1: Decomposition of the invariant positive cone.

for $j = 1, 2$ the map $g_j := \alpha_j^{-1} \circ \iota_M \circ \alpha_j$ is a Hodge isometry. Moreover, since $n = 2$ and g_j preserves the positive cone of X_j, it is a monodromy operator by Lemma

1.4.7. Furthermore, g_j fixes a Kähler class in $\mathcal{C}_j$, and by the Global Torelli theorem, there exist non-symplectic involutions $i_j : X_j \to X_j$ with $i_j^* = g_j$. We assume that there is an isomorphism $f : (X_1, i_1) \to (X_2, i_2)$ and consider

$$\psi := \alpha_1 \circ f^* \circ \alpha_2^{-1} : L \to L.$$

We have $\psi \in \Gamma(M)$ and denote by $\psi_M \in O(M)$ and $\psi_{M^\perp} \in O(M^\perp)$ its restrictions. Since η is generic and f^* is a Hodge isometry, we have $\psi_{M^\perp} = \pm \mathrm{id}_{M^\perp}$ and in particular $\psi \in \widetilde{O}(M^\perp)$. By Proposition 2.2.1, the isomorphism $\gamma : H_M \to H_{M^\perp}$ described in Section 2.2 conjugates $\psi_{M^\perp}|_{H_{M^\perp}}$ to $\psi_M|_{H_M}$. On the other hand, since f^* maps an invariant Kähler class to an invariant Kähler class, we have $\psi_M(\mathcal{C}_1) = \mathcal{C}_2$, and therefore ψ_M acts non-trivially on $H_M = A_M$, which gives a contradiction.

Now consider the universal deformations

$$(\pi_j, I_j) : \mathcal{X}_j \to \mathrm{Def}(X_j, i_j), \;\; j = 1, 2$$

The markings α_j define embeddings $\mathrm{Def}(X_j, i_j) \hookrightarrow \Omega_{M^\perp}$ as open neighbourhoods of η. By Lemma 4.6.6 every such neighbourhood contains a point $\eta' \notin \mathcal{D}'_M$. Let (X'_j, i'_j) be the deformation of (X_j, i_j) over η' and consider the Hodge isometry

$$g : (\alpha'_1)^{-1} \circ \alpha'_2 : H^2(X'_2, \mathbb{Z}) \to H^2(X'_1, \mathbb{Z}).$$

Clearly g maps the invariant positive cone of (X'_2, i'_2) to the invariant positive cone of (X'_1, i'_1). On the other hand, the invariant lattice $U(2)$ does not contain any elements with square -2 or -10, so the stable invariant Kähler cone is just the invariant positive cone. By the assumption $\eta' \notin \mathcal{D}'_M$ and Proposition 4.6.7, this also coincides with the invariant Kähler cone. Therefore, by the Global Torelli theorem, g is induced by an isomorphism $X'_1 \to X'_2$, and one easily sees that this is compatible with i'_1 and i'_2 (a detailed argument for this will be given in the proof of Theorem 4.9.5). Consequently, the pairs (X_1, i_1) and (X_2, i_2) are inseparable.

4.7 Kähler-type chambers

In this section we discuss the lattice-theoretic counterpart of the stable invariant Kähler cone, the Kähler-type chambers of M. We will use these in the next section to give a lattice-theoretic criterion for deformation equivalence.

Definition 4.7.1. A *Kähler-type chamber* of the lattice M is a connected component of

$$\widetilde{\mathcal{C}}_M \setminus \bigcup_{\delta \in \Delta(M)} \delta^\perp.$$

We denote the set of Kähler-type chambers of M by $\mathrm{KT}(M)$.

If (X, i) is a pair of type M and $\alpha : H^2(X, \mathbb{Z}) \to L$ an admissible marking, then we have $\alpha(\Delta^i(X)) = \Delta(M)$, and hence $\alpha(\widetilde{\mathcal{K}}^i_X)$ is a Kähler-type chamber of M.

Definition 4.7.2. The stable invariant Kähler cones of (X, i) and (Y, j) are *isometric* if there exists a parallel transport operator $g : H^2(X, \mathbb{Z}) \to H^2(Y, \mathbb{Z})$ with

$$j^* \circ g = g \circ i^* \quad \text{and} \quad g(\widetilde{\mathcal{K}}^i_X) = \widetilde{\mathcal{K}}^j_Y.$$

In this case we write $\widetilde{\mathcal{K}}^i_X \cong \widetilde{\mathcal{K}}^j_Y$.

Let Γ_M be the image of the homomorphism $\Gamma(M) \to O(M)$.

Lemma 4.7.3. *$\Gamma_M \subset O(M)$ is a finite index subgroup.*

Proof. By Lemma 2.4.2, any isometry in the finite index subgroup $\widetilde{O}(M) \subset O(M)$ can be extended to an isometry $\sigma \in \widetilde{O}(L)$. By Lemma 1.4.7, we have $\sigma \in \mathrm{Mon}(L)$ or $-\sigma \in \mathrm{Mon}(L)$. □

The group Γ_M acts on $\Delta(M)$ and therefore on the Kähler-type chambers of M. We clearly have $\widetilde{\mathcal{K}}^i_X \cong \widetilde{\mathcal{K}}^j_Y$ if and only if

$$[\alpha(\widetilde{\mathcal{K}}^i_X)] = [\beta(\widetilde{\mathcal{K}}^j_Y)] \in \mathrm{KT}(M)/\Gamma_M$$

for any and hence for all admissible markings α and β. In particular, we obtain a well-defined map

$$\begin{aligned} \rho :\ \mathcal{M}_M &\to \mathrm{KT}(M)/\Gamma_M \\ (X, i) &\mapsto [\alpha(\widetilde{\mathcal{K}}^i_X)]. \end{aligned}$$

We will later show, that the map ρ is surjective. For this, we will need the following Lemma.

Lemma 4.7.4. *Any Kähler-type chamber of M is an open subset of $\widetilde{\mathcal{C}}_M$.*

Proof. Let $\mathcal{C}_M \subset \widetilde{\mathcal{C}}_M$ be one of the two connected components, and let $\Gamma^+_M \subset \Gamma_M$ and $O^+(M_\mathbb{R}) \subset O(M_\mathbb{R})$ be the subgroups preserving $\mathcal{C}_M$. The group $O^+(M_\mathbb{R})$ acts transitively on

$$\mathbb{H} := \{x \in \mathcal{C}_M : (x, x) = 1\}$$

and the stabilizer of $x \in \mathbb{H}$ is the compact group $O(x^\perp)$. By [Wol67, Lemma 3.1.1], the action of the discrete subgroup $\Gamma^+_M \subset O^+(M_\mathbb{R})$ on

$$\mathbb{H} \cong O^+(M_\mathbb{R})/O(x^\perp)$$

is properly discontinuous. This implies that for $\delta \in \Delta(M)$, the Γ^+_M orbit of the closed subset $\delta^\perp \subset \mathbb{H}$ is closed. Since $\Gamma^+_M \subset O(M)$ is a finite index subgroup, there is a finite number of orbits. □

4.8 Deformation equivalence

The goal of this section is to show that two pairs (X, i) and (Y, j) of type M are deformation equivalent if and only if their stable invariant Kähler cones are isometric. Moreover, we show that every Kähler-type chamber can be realized as the stable invariant Kähler cone of some pair (X, i), and thus obtain a purely lattice-theoretic characterization of the deformation types.

Let $\mathcal{K} \in \mathrm{KT}(M)$ be a Kähler-type chamber. As a consequence of Lemma 4.7.4, there exists an integral class $h \in \mathcal{K}$. By Section 1.4.2, the component $\mathfrak{M}_L^0$ and the connected component of $\widetilde{\mathcal{C}}_M$ which contains h (and therefore $\mathcal{K}$) determine one of the connected components $\Omega_{h^\perp}^+ \subset \Omega_{h^\perp}$, such that for every

$$(X, \alpha) \in \mathfrak{M}_{h^\perp}^+ := P_0^{-1}(\Omega_{h^\perp}^+)$$

we have $\alpha^{-1}(h) \in \mathcal{C}_X$. Let $\mathfrak{M}_{h^\perp}^a \subset \mathfrak{M}_{h^\perp}^+$ be the set of marked pairs (X, α) such that $\alpha^{-1}(h)$ is ample.

Lemma 4.8.1. *$\mathfrak{M}_{h^\perp}^a \subset \mathfrak{M}_{h^\perp}^+$ is open.*

Proof. [Mar11, Cor. 7.3] □

Let $\Omega_{M^\perp}^+$ be the connected component of $\Omega_{M^\perp}$ which is contained in $\Omega_{h^\perp}^+$ and

$$\mathfrak{M}_{M^\perp}^+ := P_0^{-1}(\Omega_{M^\perp}^+).$$

We have $\alpha^{-1}(\mathcal{K}) \subset \mathcal{C}_X$ for every $(X, \alpha) \in \mathfrak{M}_{M^\perp}^+$. Let

$$\mathfrak{M}_{M^\perp, \mathcal{K}} := \{(X, \alpha) \in \mathfrak{M}_{M^\perp}^+ : \mathcal{K} \cap \alpha(\mathcal{K}_X) \neq \emptyset\}.$$

Lemma 4.8.2. *$\mathfrak{M}_{M^\perp, \mathcal{K}} \subset \mathfrak{M}_{M^\perp}^+$ is an open subset.*

Proof. Let $(X, \alpha) \in \mathfrak{M}_{M^\perp, \mathcal{K}}$. Since $\alpha(\mathcal{K}_X) \cap \mathcal{K} \subset \mathcal{K}$ is a non-empty open subset, there exists an integral element $h \in \mathcal{K}$ such that $\alpha^{-1}(h)$ is ample. Then

$$\mathfrak{M}_{h^\perp}^a \cap \mathfrak{M}_{M^\perp}^+ \subset \mathfrak{M}_{M^\perp, \mathcal{K}}$$

is an open neighbourhood of (X, α) by Lemma 4.8.1. □

Proposition 4.8.3. *The isometry class of the stable invariant Kähler cone is invariant under deformation.*

Proof. Let $\pi : (\mathcal{X}, I) \to S$ be a family over a connected base S. For $s \in S$ let

$$U_s = \{t \in S : \widetilde{\mathcal{K}}_{X_t}^{I_t} \cong \widetilde{\mathcal{K}}_{X_s}^{I_s}\}.$$

We claim that $U_s \subset S$ is open. Let $U \subset S$ be a contractible open neighbourhood of s and $\alpha : (R^2\pi_*\mathbb{Z})|_U \to L_U$ a trivialization such that α_s is admissible for (X_s, I_s).

Then for every $t \in U$ the marking α_t is admissible for (X_t, I_t), and we obtain a holomorphic map $\varphi : U \to \mathfrak{M}^+_{M^\perp}$, where $\mathfrak{M}^+_{M^\perp} = P_0^{-1}(\Omega^+_{M^\perp})$ is the connected component determined by $\mathcal{K} := \alpha_s(\widetilde{\mathcal{K}}^{I_s}_{X_s})$ as described above. By Lemma 4.8.2, the set

$$V := \varphi^{-1}(\mathfrak{M}_{M^\perp,\mathcal{K}}) \subset S$$

is a non-empty open neighbourhood of s. For every $t \in V$ there is a Kähler class inside $\alpha_t^{-1}(\mathcal{K})$, and since $\widetilde{\mathcal{K}}^{I_t}_{X_t}$ is determined by one invariant Kähler class, this implies $\widetilde{\mathcal{K}}^{I_t}_{X_t} = \alpha_t^{-1}(\mathcal{K})$. Hence $\alpha_t^{-1} \circ \alpha_s$ is a parallel transport operator mapping $\widetilde{\mathcal{K}}^{I_s}_{X_s}$ to $\widetilde{\mathcal{K}}^{I_t}_{X_t}$. This shows that $U_s \subset S$ is open and since $S = \bigcup_{s\in S} U_s$, we have $S = U_s$ for every $s \in S$. □

Let $h \in H^2(X,\mathbb{Z})^i \subset H^{1,1}(X,\mathbb{Z})$ and $\mathcal{L}$ be a line bundle on X with $c_1(\mathcal{L}) = h$. Then we have $\mathrm{Def}(X,i) \subset \mathrm{Def}(X,\mathcal{L})$. For $t \in \mathrm{Def}(X,i)$, let $h_t := c_1(\mathcal{L}_t)$ where $(X_t,\mathcal{L}_t)$ is the fibre over $t \in \mathrm{Def}(X,\mathcal{L})$ in the universal deformation of $(X,\mathcal{L})$.

The following Proposition gives a characterization of stably invariant ample classes which is similar to Markman's notion of stably prime exceptional classes [Mar13].

Proposition 4.8.4. *A class $h \in H^2(X,\mathbb{Z})^i$ belongs to $\widetilde{\mathcal{K}}^i_X$ if and only if there is an analytic subvariety $Z \subset \mathrm{Def}(X,i)$ of complex codimension 1 such that for $t \in \mathrm{Def}(X,i) \setminus Z$ the class h_t is an invariant ample class of (X_t, I_t).*

Proof. Let

$$\pi : \mathcal{X} \to \mathrm{Def}(X,i), \qquad I : \mathcal{X} \to \mathcal{X}$$

be the universal deformation of $(X,i) = (\pi^{-1}(0), I_0)$. We choose a trivialization $\alpha : R^2\pi_*\mathbb{Z} \to L_{\mathrm{Def}(X,i)}$ such that for every $t \in \mathrm{Def}(X,i)$, the marking α_t is admissible for (X_t, I_t). The period map defines an open embedding $\mathrm{Def}(X) \subset \Omega_L$ such that

$$\mathrm{Def}(X,i) = \mathrm{Def}(X) \cap \Omega_{M^\perp}.$$

We have $(X,\alpha_0) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$, where $\mathcal{K} := \alpha_0(\widetilde{\mathcal{K}}^i_X)$, and by Lemma 4.8.2 we can assume that $\mathrm{Def}(X,i)$ is sufficiently small such that $(X_t,\alpha_t) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$ for every $t \in \mathrm{Def}(X,i)$. As in the proof of Proposition 4.8.3, we see that $\widetilde{\mathcal{K}}^{I_t}_{X_t} = \alpha_t^{-1}(\mathcal{K})$. By Lemma 4.6.6, the subset

$$Z := \bigcup_{\delta\in\Delta_M(L)} \delta^\perp \subset \mathrm{Def}(X,i).$$

is a union of finitely many hyperplanes. For every $t \notin Z$ we have $\mathcal{K}^{I_t}_{X_t} = \alpha_t^{-1}(\mathcal{K})$ by Proposition 4.6.7. Hence if $h \in H^2(X,\mathbb{Z})^i$ belongs to $\widetilde{\mathcal{K}}^i_X$, then the class

$$h_t = \alpha_t^{-1} \circ \alpha_0(h) \in H^2(X_t,\mathbb{Z})^{I_t}$$

is ample for every $t \notin Z$.

Conversely let $h \in H^2(X,\mathbb{Z})^i$ and assume that there exists a $t \in \mathrm{Def}(X,i)$ such that h_t is ample. Then

$$h = \alpha_0^{-1} \circ \alpha_t(h_t) \in \alpha_0^{-1}(\mathcal{K}) = \widetilde{\mathcal{K}}_X^i$$

belongs to the stable invariant Kähler cone. $\square$

In the following our aim is to show that also the converse of Proposition 4.8.3 is true, that is, the isometry class of the stable invariant Kähler cone completely determines the deformation type.

Let $\mathcal{K}$ be a Kähler-type chamber of M and

$$P_{\mathcal{K}} : \mathfrak{M}_{M^\perp,\mathcal{K}} \to \Omega^+_{M^\perp}$$

be the restriction of the period map. Let $\Omega^0_{M^\perp} := \Omega^+_{M^\perp} \setminus \widetilde{\mathcal{D}}_M$. We have seen in Lemma 4.6.6 that $\Omega^0_{M^\perp} \subset \Omega^+_{M^\perp}$ is an open subset.

Lemma 4.8.5. *The image of $P_{\mathcal{K}}$ is $\Omega^0_{M^\perp}$.*

Proof. If $(X,\alpha) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$, then there exists a Kähler class x in $\alpha^{-1}(\mathcal{K}) \subset \alpha^{-1}(M)$. Since $x \notin D^\perp$ for every wall divisor $D \in \Delta(X)$ this shows $P(X,\alpha) \notin \widetilde{\mathcal{D}}_M$.

Conversely, assume that $\eta \in \Omega^+_{M^\perp} \setminus \widetilde{\mathcal{D}}_M$. By the surjectivity of the period map, there exists a marked pair $(X,\alpha) \in \mathfrak{M}^0_L$ with $P(X,\alpha) = \eta$. As noted before, since $\eta \in \Omega^+_{M^\perp}$ and therefore $(X,\alpha) \in \mathfrak{M}^+_{M^\perp}$, we have $\alpha^{-1}(\mathcal{K}) \subset \mathcal{C}_X$.

We claim that the cone $\alpha^{-1}(\mathcal{K})$ is not contained in the hyperplane $D^\perp$ for any $D \in \Delta(X)$. Indeed, since $\mathcal{K} \subset M_{\mathbb{R}}$ is open, this would imply $\delta := \alpha(D) \in M^\perp$, and therefore $\delta \in \Delta(M^\perp)$. Then $P(X,\alpha) \in \delta^\perp \subset \widetilde{\mathcal{D}}_M$ gives a contradiction.

Now it follows from Theorem 4.5.3, that $\alpha^{-1}(\mathcal{K})$ intersects a Kähler-type chamber of X. By definition, this means, that there exists an isometry $g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ and a birational model $f : X \dashrightarrow \widetilde{X}$ such that

$$\alpha^{-1}(\mathcal{K}) \cap g(f^*\mathcal{K}_{\widetilde{X}}) \neq \emptyset$$

and therefore $\widetilde{\alpha}^{-1}(\mathcal{K}) \cap \mathcal{K}_{\widetilde{X}} \neq \emptyset$, where

$$\widetilde{\alpha} := \alpha \circ g \circ f^* : H^2(\widetilde{X},\mathbb{Z}) \to L.$$

Since $g \circ f^*$ is a Hodge isometry and a parallel transport operator, we have

$$P_0(\widetilde{X},\widetilde{\alpha}) = P_0(X,\alpha) = \eta$$

and hence $(\widetilde{X},\widetilde{\alpha}) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$ is a marked pair with $P_{\mathcal{K}}(\widetilde{X},\widetilde{\alpha}) = \eta$. $\square$

Since $\mathfrak{M}_{M^\perp,\mathcal{K}} \subset \mathfrak{M}^+_{M^\perp}$ is open, the period map restricts to a local isomorphism $P_{\mathcal{K}} : \mathfrak{M}_{M^\perp,\mathcal{K}} \to \Omega^0_{M^\perp}$. We now want to use the path-connectedness of $\Omega^0_{M^\perp}$ to show that $\mathfrak{M}_{M^\perp,\mathcal{K}}$ is path-connected. We will then define a family of involutions over $\mathfrak{M}_{M^\perp,\mathcal{K}}$ containing any pair (X,i) with $\rho(X,i) = [\mathcal{K}]$.

Lemma 4.8.6. *The space $\Omega^0_{M^\perp}$ is path-connected.*

Proof. Let $\eta_1, \eta_2 \in \Omega^0_{M^\perp} \subset \Omega^+_{M^\perp}$ and $\gamma : [0,1] \to \Omega^+_{M^\perp}$ be a path connecting η_1 and η_2. By Lemma 4.6.6, for any $t \in [0,1]$, there exists a path-connected open neighbourhood $U_t \subset \Omega^+_{M^\perp}$ of $\gamma(t)$ which intersects only finitely many hyperplanes $\delta^\perp$ for $\delta \in \Delta(M^\perp)$. Let $V_1, \ldots, V_k$ be a finite subcovering of $\{U_t\}$ such that $\eta_1 \in V_1$, $\eta_2 \in V_k$ and

$$V_i \cap V_{i+1} \neq \emptyset, \qquad i = 0, \ldots, k-1.$$

For any i, the set $V_i \setminus \widetilde{\mathcal{D}}_M$ is the complement in V_i of finitely many hyperplanes of real codimension 2 and therefore path-connected. Since $V_i \cap V_{i+1} \subset \Omega^+_{M^\perp}$ is open, we have $V_i \cap V_{i+1} \cap \Omega^0_{M^\perp} \neq \emptyset$, which shows the claim. $\square$

Locally, paths in $\Omega^0_{M^\perp}$ can be lifted to paths in $\mathfrak{M}_{M^\perp,\mathcal{K}}$ using the Local Torelli theorem. To connect these paths in $\mathfrak{M}_{M^\perp,\mathcal{K}}$, we will need a dense subset of points which are unique in their fibers with respect to $P_\mathcal{K}$. Let

$$\Omega'_{M^\perp} := \Omega_{M^\perp} \setminus \bigcup_{\delta \in L \setminus M} \delta^\perp$$

and

$$\mathfrak{M}'_{M^\perp,\mathcal{K}} := P_\mathcal{K}^{-1}(\Omega'_{M^\perp}).$$

For $(X,\alpha) \in \mathfrak{M}'_{M^\perp,\mathcal{K}}$ we have $\alpha(H^{1,1}(X,\mathbb{Z})) = M$ and therefore $\alpha(\mathcal{K}_X) = \mathcal{K}$.

Lemma 4.8.7. *If $(X,\alpha) \in \mathfrak{M}'_{M^\perp,\mathcal{K}}$, then*

$$P_\mathcal{K}^{-1}(P_\mathcal{K}(X,\alpha)) = \{(X,\alpha)\}.$$

Proof. For $i = 1,2$, let $(X_i,\alpha_i) \in \mathfrak{M}'_{M^\perp,\mathcal{K}}$ with $P_\mathcal{K}(X_1,\alpha_1) = P_\mathcal{K}(X_2,\alpha_2)$. Then $\alpha_2^{-1} \circ \alpha_1 : H^2(X_1,\mathbb{Z}) \to H^2(X_2,\mathbb{Z})$ is a Hodge isometry and a parallel transport operator that maps $\mathcal{K}_{X_1}$ onto $\mathcal{K}_{X_2}$. By the Global Torelli theorem, $\alpha_2^{-1} \circ \alpha_1$ is induced by an isomorphism $f : X_2 \to X_1$, which defines an isomorphism $(X_1,\alpha_1) \cong (X_2,\alpha_2)$ of marked pairs. $\square$

The following Proposition is a generalization of [Mar13, Cor. 5.11], which contains the same statement for a rank 1 lattice $M = \mathbb{Z}h$ with $(h,h) > 0$. In this case $\mathcal{K}$ is the ray $\mathbb{R}_{>0} \cdot h$ and $\mathfrak{M}_{M^\perp,\mathcal{K}} = \mathfrak{M}^a_{h^\perp}$. We will use the same idea for the proof.

Proposition 4.8.8. *$\mathfrak{M}_{M^\perp,\mathcal{K}}$ is path-connected.*

Proof. By the Local Torelli theorem and Lemma 4.8.2, the surjective map

$$P_\mathcal{K} : \mathfrak{M}_{M^\perp,\mathcal{K}} \to \Omega^0_{M^\perp}$$

is a local isomorphism. Let

$$(X_0,\alpha_0), (X_1,\alpha_1) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$$

and $\eta_i := P_{\mathcal{K}}(X_i, \alpha_i)$. Let $\gamma : [0,1] \to \Omega^0_{M^\perp}$ be a continuous path with $\gamma(i) = \eta_i$. It follows from the proof of Lemma 4.8.6 that γ can be chosen sufficiently generic such that

$$T := \gamma^{-1}(\Omega'_{M^\perp}) \subset [0,1]$$

is dense. For every $s \in P_{\mathcal{K}}^{-1}(\gamma([0,1]))$ let $U_s \subset \mathfrak{M}_{M^\perp,\mathcal{K}}$ be a path-connected open neighbourhood of s which is mapped isomorphically onto an open subset of $\Omega^0_{M^\perp}$. Then the sets $P_{\mathcal{K}}(U_s)$ form an open covering of $\gamma([0,1])$, and we choose a finite subcovering

$$V_i = P_{\mathcal{K}}(U_{s_i}),\ i = 0, \ldots, n+1.$$

We can assume that

$$p_0 := (X_0, \alpha_0) \in U_{s_0},\ p_{n+1} := (X_1, \alpha_1) \in U_{s_n}$$

and that $V_{i-1} \cap V_i \neq \emptyset$ for $i = 1, \ldots, n$. Let $t_0 := 0,\ t_{n+1} := 1$,

$$t_i \in \gamma^{-1}(V_{i-1} \cap V_i) \cap T, \qquad i = 1, \ldots, n,$$

and $p_i \in \mathfrak{M}'_{M^\perp,\mathcal{K}}$ be the unique element in $P_{\mathcal{K}}^{-1}(\gamma(t_i))$. Using the isomorphism $P_{\mathcal{K}}|_{U_{s_i}} : U_{s_i} \to V_i$, the path $\gamma|_{[t_i,t_{i+1}]}$ can be lifted to a path connecting p_i with p_{i+1}. □

Proposition 4.8.9. *For $(X, \alpha) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$, there exists a non-symplectic involution $i : X \to X$ with $i^* = \alpha^{-1} \circ \iota_M \circ \alpha$. In particular, (X, i) is of type M.*

Proof. Since M is admissible, we have $\iota_M \in \mathrm{Mon}(L)$, and therefore

$$g := \alpha^{-1} \circ \iota_M \circ \alpha : H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is a monodromy operator. Furthermore, from $P(X, \alpha) \in \Omega_{M^\perp}$ we obtain

$$\alpha(H^{2,0}(X)) \subset M^\perp_{\mathbb{C}}$$

and hence that $g(\omega) = -\omega$. In particular, g is a Hodge isometry. Finally, g acts trivially on the chamber $\alpha^{-1}(\mathcal{K})$, which by assumption contains a Kähler class. By the Global Torelli theorem, there exists an automorphism $i : X \to X$ with $i^* = g$. We have $i^* \circ i^* = \mathrm{id}_{H^2(X,\mathbb{Z})}$, which by Theorem 4.1.3 shows that i is an involution. □

Theorem 4.8.10. *Let $(X_0, i_0), (X_1, i_1)$ be two pairs of type M with isometric stable invariant Kähler cones. Then (X_0, i_0) and (X_1, i_1) are deformation equivalent.*

Proof. Let $g : H^2(X_1, \mathbb{Z}) \to H^2(X_0, \mathbb{Z})$ be a parallel transport operator with

$$i_0^* \circ g = g \circ i_1^*, \qquad g(\widetilde{\mathcal{K}}^{i_1}_{X_1}) = \widetilde{\mathcal{K}}^{i_0}_{X_0}.$$

Let $\alpha_0 : H^2(X_0, \mathbb{Z}) \to L$ be an admissible marking of (X, i_0) and let

$$\alpha_1 := \alpha_0 \circ g : H^2(X_1, \mathbb{Z}) \to L.$$

For $j = 0, 1$, we have $(X_j, \alpha_j) \in \mathfrak{M}_{M^\perp, \mathcal{K}}$, where

$$\mathcal{K} := \alpha_0(\widetilde{\mathcal{K}}^{i_0}_{X_0}) = \alpha_1(\widetilde{\mathcal{K}}^{i_1}_{X_1}).$$

For every $(X, \alpha_X) \in \mathfrak{M}_{M^\perp, \mathcal{K}}$ there exists an involution $i : X \to X$ such that

$$i^* = \alpha_X^{-1} \circ \iota_M \circ \alpha_X$$

by Proposition 4.8.9, which is unique by Theorem 4.1.3. These involutions fit into a holomorphic family $(\mathcal{X}, I) \to \mathfrak{M}_{M^\perp, \mathcal{K}}$. Indeed, let $U \subset \mathfrak{M}_{M^\perp, \mathcal{K}}$ be a contractible open neighbourhood of (X, α_X) and

$$\pi_U : \mathcal{X}_U \to U,$$
$$\alpha : (R^2\pi_*\mathbb{Z})|_U \to L_U$$

be the universal family of marked manifolds over U. The involution $I_U : \mathcal{X}_U \to \mathcal{X}_U$ which is defined on each fiber as above, is holomorphic, since it coincides with the universal deformation of (X, i). If U, V are two such sets, we can glue $\mathcal{X}_U$ and $\mathcal{X}_V$ over $U \cap V$ and obtain a global family $\mathcal{X} \to \mathfrak{M}_{M^\perp, \mathcal{K}}$, since marked pairs do not admit non-trivial automorphisms by Theorem 4.1.3. Since the involutions I_U and I_V coincide over $U \cap V$, we obtain a holomorphic involution $I : \mathcal{X} \to \mathcal{X}$ containing (X_0, i_0) and (X_1, i_1). □

Theorem 4.8.11. *The map $\rho : \mathcal{M}_M \to \mathrm{KT}(M)/\Gamma_M$ induces a bijection between deformation types of pairs of type M and $\mathrm{KT}(M)/\Gamma_M$.*

Proof. It remains to show surjectivity. Let $\mathcal{K} \in \mathrm{KT}(M)$ be a Kähler-type chamber. By Lemma 4.8.5, the set $\mathfrak{M}_{M^\perp, \mathcal{K}}$ is non-empty, and by Proposition 4.8.9, for any $(X, \alpha) \in \mathfrak{M}_{M^\perp, \mathcal{K}}$ there exists an involution $i : X \to X$ with $i^* = \alpha^{-1} \circ \iota_M \circ \alpha$. The isometry i^* acts trivially on $\alpha^{-1}(\mathcal{K})$, which implies $\alpha^{-1}(\mathcal{K}) \cap \mathcal{K}^i_X = \alpha^{-1}(\mathcal{K}) \cap \mathcal{K}_X$. Since $(X, \alpha) \in \mathfrak{M}_{M^\perp, \mathcal{K}}$, this intersection is non-empty, and since $\mathcal{K}^i_X \subset \widetilde{\mathcal{K}}^i_X$, this shows $\alpha(\widetilde{\mathcal{K}}^i_X) = \mathcal{K}$. □

Remark 4.8.12. Recall that in general the definition of type M and of admissible markings depends on the choice of the connected component $\mathfrak{M}^0_L$ of $\mathfrak{M}_L$. As a consequence, also the bijection ρ depends on this choice. However, by Remark 4.2.3 it is clear that another choice of a connected component gives a different but equivalent bijection. Moreover, if $n = 2$ or $n - 1$ is a prime power, then the map ρ does not depend on this choice.

Example 4.8.13. We apply Theorem 4.8.11 to an example by Ohashi–Wandel [OW13]. Let $\pi : S \to \mathbb{P}^2$ be a K3 surface which is a double plane branched over a smooth sextic. Let $i : S \to S$ be the covering involution and $i^{[2]} : S^{[2]} \to S^{[2]}$ the natural involution. The fixed locus of $i^{[2]}$ contains the plane

$$P := \{[s, i(s)] \in S^{[2]} : s \in S\} \cong S/i \cong \mathbb{P}^2.$$

Ohashi and Wandel consider the Mukai flop $\psi : X \dashrightarrow S^{[2]}$ obtained by replacing P by the dual projective plane $P^* = |\mathcal{O}_P(1)|$ and show, that the induced birational involution

$$j := \psi^{-1} \circ i^{[2]} \circ \psi : X \to X$$

is biregular. The invariant lattice of i is isometric to $\langle 2 \rangle$ and therefore that of the natural involution $i^{[2]}$ is given by

$$H^2(S^{[2]}, \mathbb{Z})^{i^{[2]}} = \mathbb{Z}h \oplus \mathbb{Z}e \cong \langle 2 \rangle \oplus \langle -2 \rangle,$$

where e is half the class of the exceptional divisor and h is the image of a primitive invariant ample class on S under the natural map $\mathrm{NS}(S) \to \mathrm{NS}(S^{[2]})$. Hence $(S^{[2]}, i^{[2]})$ is of type

$$M := \varepsilon(\langle 2 \rangle) \oplus \langle -2 \rangle \subset L,$$

where

$$\varepsilon : L_{K3} \hookrightarrow L = L_{K3} \oplus \langle -2 \rangle$$

is the natural inclusion. The fundamental exceptional chamber of $S^{[2]}$ is divided into two chambers $\mathcal{K}_1$ and $\mathcal{K}_2$ by the wall $D^\perp$, where $D := 2h + 3e \in \Delta(S^{[2]})$ is a -10-class with divisor 2.

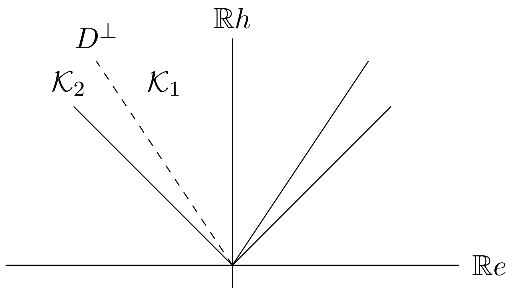


Figure 4.2: Decomposition of the positive cone.

By [BM13, Lemma 13.3], the (invariant) Kähler cone of $S^{[2]}$ is equal to $\mathcal{K}_1$. Since the flop X is not isomorphic to $S^{[2]}$, this implies $\mathcal{K}_2 = \psi^*(\mathcal{K}_X)$. Ohashi and Wandel show that every pair of type M can be deformed into $(S^{[2]}, i^{[2]})$ or into (X, j), see [OW13, Cor. 2.11].

Since $\mathcal{K}_1$ and $\mathcal{K}_2$ are not isometric, the pairs $(S^{[2]}, i^{[2]})$ and (X, j) are not deformation equivalent. This answers a question from [OW13]. There are exactly two deformation types of pairs of type M, one of which contains all natural involutions.

4.9 Moduli spaces

By Theorem 4.8.11, the deformation equivalence classes of $\mathcal{M}_M$ are

$$\mathcal{M}_{M,\mathcal{K}} := \{(X,i) \in \mathcal{M}_M : \rho(X,i) = [\mathcal{K}]\},$$

where $[\mathcal{K}] \in \mathrm{KT}(M)/\Gamma_M$. In this section, we want to replace the period map P_M by a finer period map $P_{M,\mathcal{K}}$ which maps $\mathcal{M}_{M,\mathcal{K}}$ generically injectively onto a quasi-projective variety. For the rest of this section, we fix a representative $\mathcal{K}$ of $[\mathcal{K}]$ and denote by $\Omega^+_{M^\perp}$ the connected component determined by $\mathfrak{M}^0_L$ and $\mathcal{K}$.

Definition 4.9.1. Let $(X,i) \in \mathcal{M}_{M,\mathcal{K}}$. A marking $\alpha : H^2(X,\mathbb{Z}) \to L$ is called *admissible for* $\mathcal{K}$, if it is admissible for M and furthermore satisfies $\alpha(\widetilde{\mathcal{K}}^i_X) = \mathcal{K}$.

By definition of $\mathcal{M}_{M,\mathcal{K}}$, for any pair $(X,i) \in \mathcal{M}_{M,\mathcal{K}}$ there exists a marking which is admissible for $\mathcal{K}$. Moreover any two such markings differ by an element of

$$\Gamma(\mathcal{K}) := \{\sigma \in \Gamma(M) : \sigma(\mathcal{K}) = \mathcal{K}\} \subset O(L).$$

Let $\Gamma_{M^\perp,\mathcal{K}}$ be the image of the restriction homomorphism $\Gamma(\mathcal{K}) \to O(M^\perp)$. Since

$$\Gamma(\mathcal{K}) \subset \mathrm{Mon}(L) \subset O^+(L),$$

and since its image in $O(M)$ is contained in $O^+(M)$, we have

$$\Gamma_{M^\perp,\mathcal{K}} \subset O^+(M^\perp),$$

where $O^+(M^\perp)$ is the subgroup of isometries with real spinor norm $+1$, or equivalently, the subgroup of isometries preserving $\Omega^+_{M^\perp}$.

Proposition 4.9.2. *$\Gamma_{M^\perp,\mathcal{K}}$ is a finite index subgroup of $O^+(M^\perp)$.*

Proof. As before, any isometry in the finite index subgroup $\widetilde{O}^+(M^\perp)$ extends to an isometry in $\sigma \in \widetilde{O}(L)$ acting trivially on M, and by Lemma 1.4.7 we have $\sigma \in \mathrm{Mon}(L)$ and thus $\sigma \in \Gamma(K)$. □

The quotient $\Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}}$ is a quasi-projective variety, and we have a well-defined map

$$\begin{aligned} P_{M,\mathcal{K}} : \mathcal{M}_{M,\mathcal{K}} &\to \Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}} \\ (X,i) &\mapsto P(X,\alpha), \end{aligned} \tag{4.6}$$

where $\alpha : H^2(X,\mathbb{Z}) \to L$ is any marking which is admissible for $\mathcal{K}$.

Assume that $\pi : (\mathcal{X}, I) \to S$ is a deformation of $(X,i) = \pi^{-1}(0)$. Let $U \subset S$ be a contractible open neighbourhood of 0 and $\alpha : (R^2\pi_*\mathbb{Z})|_U \to L_U$ a trivialization such that α_0 is admissible for $\mathcal{K}$. Then for every $t \in U$ the marking α_t is admissible for $\mathcal{K}$, as shown in the proof of Proposition 4.8.3. Since the ordinary period map is holomorphic, this shows that the induced map

$$\begin{aligned} S &\rightarrow \Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}} \\ t &\mapsto P_{M,\mathcal{K}}(X_t, I_t) \end{aligned}$$

is holomorphic.

We will see that the map $P_{M,\mathcal{K}}$ is generically injective. However, as seen in Example 4.6.8, in general $\mathcal{M}_{M,\mathcal{K}}$ does not admit a structure as a Hausdorff moduli space, and in particular $P_{M,\mathcal{K}}$ need not be injective. We therefore restrict to a certain class of pairs (X, i) in order to obtain a quasi-projective (and in particular Hausdorff) moduli space.

Definition 4.9.3. A pair (X, i) of type M is called *simple*, if $\widetilde{\mathcal{K}}^i_X = \mathcal{K}^i_X$.

Let

$$\Delta(\mathcal{K}) := \{\delta \in \Delta_M(L) : \delta^\perp \cap \mathcal{K} \neq \emptyset\}.$$

Since $\Delta(\mathcal{K}) \subset \Delta_M(L)$, it follows from Lemma 4.6.6, that the collection of hyperplanes

$$\{\delta^\perp \subset \Omega^+_{M^\perp},\ \delta \in \Delta(\mathcal{K})\}$$

is locally finite and thus their union

$$\widetilde{\mathcal{D}}_{\mathcal{K}} := \bigcup_{\delta \in \Delta(\mathcal{K})} \delta^\perp \subset \Omega^+_{M^\perp}$$

is closed. Furthermore, $\widetilde{\mathcal{D}}_{\mathcal{K}}$ is invariant under $\Gamma_{M^\perp,\mathcal{K}}$ and the quotient

$$\mathcal{D}_{\mathcal{K}} := \widetilde{\mathcal{D}}_{\mathcal{K}}/\Gamma_{M^\perp,\mathcal{K}} \subset \Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}}$$

is Zariski-closed. Hence

$$\mathcal{M}^0_{M,\mathcal{K}} := (\Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}}) \setminus (\mathcal{D}_M \cup \mathcal{D}_{\mathcal{K}})$$

is a quasi-projective variety.

Proposition 4.9.4. *A pair (X, i) of deformation type $\mathcal{K}$ is simple if and only if $P_{M,\mathcal{K}}(X, i) \notin \mathcal{D}_{\mathcal{K}}$.*

Proof. Let $\alpha : H^2(X, \mathbb{Z}) \rightarrow L$ be a marking which is admissible for $\mathcal{K}$. If (X, i) is not simple, then in the proof of Proposition 4.6.7 it was shown that there exists an element $\delta \in \Delta_M(L)$ such that $P(X, \alpha) \in \delta^\perp$ and $D^\perp \cap \widetilde{\mathcal{K}}^i_X \neq \emptyset$, where $D := \alpha^{-1}(\delta)$. Thus we have in fact $\delta \in \Delta(\mathcal{K})$, which shows one implication.

Conversely, assume that $P(X, \alpha) \in \delta^\perp$, where $\delta \in \Delta(\mathcal{K})$. Then $D := \alpha^{-1}(\delta) \in \Delta(X)$ is a wall divisor with $D^\perp \cap \widetilde{\mathcal{K}}^i_X \neq \emptyset$, which implies that (X, i) is not simple. ☐

Theorem 4.9.5. *$\mathcal{M}^0_{M,\mathcal{K}}$ is a coarse moduli space for simple pairs of type M and of deformation type $[\mathcal{K}]$.*

Proof. Let $\eta \in \Omega^0_{M^\perp} = \Omega^+_{M^\perp} \setminus \widetilde{\mathcal{D}}_M$. By Lemma 4.8.5 there exists a marked pair $(X, \alpha) \in \mathfrak{M}_{M^\perp,\mathcal{K}}$ with $P(X, \alpha) = \eta$, and by Proposition 4.8.9, there exists a non-symplectic involution $i : X \to X$ with $i^* = \alpha^{-1} \circ \iota_M \circ \alpha$. As in the proof of Theorem 4.8.11 one sees that $\alpha(\widetilde{\mathcal{K}}^i_X) = \mathcal{K}$. Together with Proposition 4.9.4, this shows that the period map $P_{M,\mathcal{K}}$ restricts to a surjective map

$$P_{M,\mathcal{K}} : \{(X, i) \in \mathcal{M}_{M,\mathcal{K}} : (X, i) \text{ is simple}\} \to \mathcal{M}^0_{M,\mathcal{K}}.$$

It remains to show that this map is injective. Assume that $(X_0, i_0), (X_1, i_1)$ are two simple pairs with $P_{M,\mathcal{K}}(X_0, i_0) = P_{M,\mathcal{K}}(X_1, i_1)$. Let

$$\alpha_j : H^2(X_j, \mathbb{Z}) \to L, \quad j = 0, 1$$

be markings that are admissible for $\mathcal{K}$. By assumption, there exists an isometry $\tau \in \Gamma(\mathcal{K})$ such that $\tau(P_0(X_1, \alpha_1)) = P_0(X_0, \alpha_0)$. This means, that

$$g := \alpha_0^{-1} \circ \tau \circ \alpha_1 : H^2(X_1, \mathbb{Z}) \to H^2(X_0, \mathbb{Z})$$

is a Hodge isometry. Since $(X_0, \alpha_0), (X_1, \alpha_1) \in \mathfrak{M}^0_L$ and $\tau \in \mathrm{Mon}(L)$, it is a parallel transport operator. Furthermore, we have

$$g(\mathcal{K}^{i_1}_{X_1}) = g(\widetilde{\mathcal{K}}^{i_1}_{X_1}) = \widetilde{\mathcal{K}}^{i_0}_{X_0} = \mathcal{K}^{i_0}_{X_0}.$$

Since the invariant Kähler cones are non-empty, g maps a Kähler class to a Kähler class, and by the Global Torelli theorem, there exists an isomorphism $f : X_0 \to X_1$ with $f^* = g$. Moreover,

$$\begin{aligned}(i_1 \circ f)^* &= f^* \circ i_1^* = \alpha_0^{-1} \circ \tau \circ \alpha_1 \circ i_1^* = \alpha_0^{-1} \circ \tau \circ \iota_M \circ \alpha_1 \\ &= \alpha_0^{-1} \circ \iota_M \circ \tau \circ \alpha_1 = i_0^* \circ \alpha_0^{-1} \circ \tau \circ \alpha_1 = i_0^* \circ f^* \\ &= (f \circ i_0)^*,\end{aligned}$$

which by Theorem 4.1.3 implies that $i_1 \circ f = f \circ i_0$ and hence that

$$f : (X_0, i_0) \xrightarrow{\sim} (X_1, i_1). \qquad \square$$

Corollary 4.9.6. *Suppose that (X_1, i_1) and (X_2, i_2) are two non-isomorphic pairs of type M and deformation type $\mathcal{K}$ with*

$$P_{M,\mathcal{K}}(X_1, i_1) = P_{M,\mathcal{K}}(X_2, i_2).$$

Then (X_1, i_1) and (X_2, i_2) are inseparable.

Proof. By the assumption $P_{M,\mathcal{K}}(X_1, i_1) = P_{M,\mathcal{K}}(X_2, i_2)$, there exist admissible markings $\alpha_1 : H^2(X_1, \mathbb{Z}) \to L$ and $\alpha_2 : H^2(X_2, \mathbb{Z}) \to L$ with

$$\eta := P_0(X_1, \alpha_1) = P_0(X_2, \alpha_2).$$

These induce embeddings

$$\mathrm{Def}(X_1, i_1),\, \mathrm{Def}(X_2, i_2) \subset \Omega_{M^\perp}, \qquad j = 1, 2$$

as open neighborhoods of η. Let η' be any point inside the open subset

$$(\mathrm{Def}(X_1, i_1) \cap \mathrm{Def}(X_2, i_2)) \setminus \widetilde{\mathcal{D}}_{\mathcal{K}}.$$

By Theorem 4.9.5, there is a unique pair (X', i') which is in the fibre over η' in the universal deformation of both (X_1, i_1) and (X_2, i_2). $\square$

The following example shows that the groups $\Gamma_{M^\perp,\mathcal{K}}$ can be different for different deformation classes $\mathcal{M}_{M,\mathcal{K}}$ of $\mathcal{M}_M$.

Example 4.9.7. We again consider a double plane $\pi : S \to \mathbb{P}^2$, this time branched over a generic sextic curve $C \subset \mathbb{P}^2$ with two nodes $Q, Q' \in C$. Let $i : S \to S$ be the covering involution and $i^{[2]} : S^{[2]} \to S^{[2]}$ the natural involution. The invariant lattice of i is generated by the class c of a genus 2 curve which is the pullback of a line, and the classes d, d' of the exceptional divisors obtained by blowing up Q and Q'. Therefore, the invariant lattice of the natural involution is given by

$$\begin{aligned} H^2(S^{[2]}, \mathbb{Z})^{i^{[2]}} = &= \mathbb{Z}c \oplus \mathbb{Z}d \oplus \mathbb{Z}d' \oplus \mathbb{Z}e \\ &\cong \langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle, \end{aligned}$$

where e is half the class of the exceptional divisor on $S^{[2]}$, and $H^2(S, \mathbb{Z})$ is identified with its image in $H^2(S^{[2]}, \mathbb{Z})$. Hence $(S^{[2]}, i^{[2]})$ is of type M, where

$$M := \varepsilon(\langle 2 \rangle \oplus \langle -2 \rangle \oplus \langle -2 \rangle) \oplus \langle -2 \rangle \subset L$$

and

$$\varepsilon : L_{K3} \hookrightarrow L = L_{K3} \oplus \langle -2 \rangle$$

is the natural inclusion. We will implicitly identify $\mathrm{NS}(S^{[2]}) = H^2(S^{[2]}, \mathbb{Z})^{i^{[2]}}$ with M. Consider the -2-classes

$$\begin{aligned} \delta_1 &= -d, \\ \delta_2 &= -d', \\ \delta_3 &= -e, \\ \delta_4 &= c + d + d', \\ \delta_5 &= c + d + e, \\ \delta_6 &= c + d' + e. \end{aligned}$$

The polyhedron

$$P := \{x \in \mathcal{C}_X : (\delta_i, x) \geq 0 \text{ for } i = 1, \ldots, 6\}/\mathbb{R}_{>0}$$

is the convex hull of

$$p_0 = c,\ p_1 = c + d,\ p_1' = c + d',\ p_2 = c + e,\ p_3 = 2c + d + d' + e.$$

The only non-trivial isometry $\sigma \in \Gamma_M$ preserving P is the involution given by $d \mapsto d'$. Indeed, such an isometry acts on the set of p_i, which are uniquely determined as primitive integral representatives of the vertices of P. We have

$$(p_0, p_0) = (p_3, p_3) = 2$$

and

$$(p_1, p_1) = (p_1', p_1') = (p_2, p_2) = 0.$$

Since σ extends to L, we have $\sigma(e + 2M) = e + 2M$, which shows the claim. In particular there is no stable isometry preserving P and hence there is no $\delta \in M$ with $(\delta, \delta) = -2$ such that $\delta^\perp$ meets the interior of P. Therefore, the interior of P is equal to $\mathcal{E}/\mathbb{R}_{>0}$, where $\mathcal{E}$ is an exceptional chamber of $S^{[2]}$.

Let $\delta \in M$ be an element with $(\delta, \delta) = -10$ and $\operatorname{div}_L(\delta) = 2$. We assume that $(\delta, c) \geq 0$ and that the hyperplane $\delta^\perp$ meets the interior of P. Then there exists a $p \in \{p_1, p_1', p_2, p_3\}$ with $(\delta, p) < 0$. A simple calculation shows that

$$\delta \in \{2d - e,\ 2d' - e,\ 2c + 3e,\ 2c + 2d + 2d' + e\}.$$

The corresponding hyperplanes divide P into 6 polyhedra with vertices

$$\begin{aligned}
P_1 &= \{p_0, q_1, q_1', q_2, q_3\}\\
P_2 &= \{p_2, q_1, q_1', q_2, q_3\}\\
P_3 &= \{p_0, p_1, q_1, q_2\}\\
P_3' &= \{p_0, p_1', q_1', q_2\}\\
P_4 &= \{p_0, p_1, p_1', q_2\}\\
P_5 &= \{p_1, p_1', p_3, q_2\}
\end{aligned}$$

where

$$q_1 = 3c + d + 2e,\ q_1' = 3c + d' + 2e,\ q_2 = 3c + d + d' + 2e,\ q_3 = 3c + 2e.$$

We call two such polyhedra *adjacent*, if they have a common face $\delta^\perp$ for some -10-class δ.

The involution σ maps P_3 to P_3' and fixes the other P_i. Hence the moduli space consists of four components which are Zariski-open subsets

$$\mathcal{M}^0_{M,P_i} \subset \Omega_{M^\perp}/\Gamma_{M^\perp},\ i \neq 3$$

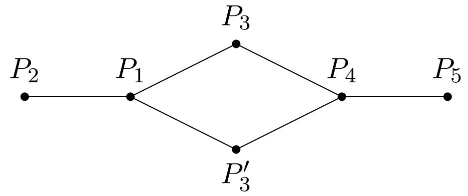


Figure 4.3: Adjacency of Kähler-type chambers

and one component

$$\mathcal{M}^0_{M,P_3} \subset \Omega_{M^\perp}/\widetilde{O}(M^\perp).$$

Moreover, $\widetilde{O}(M^\perp) \subset \Gamma_{M^\perp}$ is an index 2 subgroup and the projection map

$$\Omega_{M^\perp}/\widetilde{O}(M^\perp) \to \Omega_{M^\perp}/\Gamma_{M^\perp}$$

is a double cover.

We now want to interpret this double cover geometrically. By [BM13, Lemma 13.3], the Kähler cone of $S^{[2]}$ corresponds to the polyhedron P_1, which is adjacent to P_2, P_3 and P_3'. On the other hand, the fixed locus of $i^{[2]}$ contains the three planes:

(i) the symmetric products $D^{(2)}, (D')^{(2)} \cong (\mathbb{P}^1)^{(2)} \cong \mathbb{P}^2,\ i = 1, 2$, where $D, D' \subset S$ are the exceptional divisors of the blow-ups of Q, Q',

(ii) the closure P of $\{[s, i(s)] \in S^{[2]} : s \in S \setminus S^i\}$ which is isomorphic to $S/i \cong \mathbb{P}^2$.

As in Example 4.8.13, the fact that the planes are contained in the fixed locus implies that the induced involutions on the corresponding flops X, X', Y are biregular. Since a flop corresponds to a reflection in a -10-wall [MW14, Rem. 5.2], the Kähler cones of the flops X and X' correspond to P_3 and P_3', and that of Y to P_2. Now we have:

(i) (X, j) and (X', j') are not isomorphic: by Example 4.3.1, any deformation of $(S^{[2]}, i^{[2]})$ is induced by (S, i) and hence by a deformation of C such that the two nodes remain nodes. Therefore the genericity of C implies the genericity of $\eta \in \Omega_{M^\perp}$. Now the claim can be seen as in Example 4.6.8: the only possible isometry is given by σ, which does not extend to an isometry of L acting trivially on $M^\perp$, and is therefore not a Hodge isometry.

(ii) (X, j) and (X', j') are deformation equivalent: Consider a deformation of $C \subset \mathbb{P}^2$ into itself, such that the two nodes remain nodes and Q moves to Q'. This induces a deformation of the flops, which exchanges (X, j) and (X', j').

4.10 $K3^{[2]}$-type

In this section we want to give slightly stronger results for $K3^{[2]}$-type manifolds. Let $D \in \Delta(X)$ be a class with $(D, D) = -2$ and $(D, h) > 0$ for an ample class h. Then D or $2D$ is effective by [Mar13, Thm. 1.11]. On the other hand, if $E \in \mathcal{P}_X$ is a prime exceptional divisor, then we have $E = D$ or $E = 2D$ for some -2-class $D \in \Delta(X)$ by [Mar13, Thm. 1.2]. We denote by $\Delta_{\mathcal{P}}(X) \subset \Delta(X)$ the set of -2-classes D such that D or $2D$ is prime exceptional. Then the fundamental exceptional chamber $\mathcal{FE}_X$ is a connected component of

$$\mathcal{C}_X \setminus \bigcup_{D \in \Delta_{\mathcal{P}}(X)} D^{\perp}.$$

Definition 4.10.1. Let $i : X \to X$ be a non-symplectic involution,

(i) The *invariant exceptional chambers* of $\mathcal{C}_X^i$ are the connected components of

$$\mathcal{C}_X^i \setminus \bigcup_{D \in \Delta_{\mathcal{P}}(X)} D^{\perp}.$$

(ii) The *fundamental invariant exceptional chamber* $\mathcal{FE}_X^i$ is the invariant exceptional chamber containing the invariant Kähler cone.

Note that we remove hyperplanes orthogonal to all prime exceptional divisors, not just invariant ones. Thus the invariant exceptional chambers are simply the non-empty intersections of $\mathcal{C}_X^i$ with the exceptional chambers of $\mathcal{C}_X$.

The proof of the following Lemma is similar to the proof of a corresponding result for K3 surfaces and arbitrary finite groups by Oguiso and Sakurai [OS01, Lemma 1.3].

Lemma 4.10.2. *Let X be of $K3^{[2]}$-type and $i : X \to X$ a non-symplectic involution. The group*

$$\Gamma_i := \{g \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X) : g \circ i^* = i^* \circ g\}$$

acts transitively on the set of invariant exceptional chambers.

Proof. We will show that for any $x \in \mathcal{C}_X^i$ that belongs to some invariant exceptional chamber, there exists an isometry $\tau \in \Gamma_i$ with $\tau(x) \in \mathcal{FE}_X^i$, that is, $(\tau(x), D) > 0$ for any $D \in \Delta_{\mathcal{P}}(X)$. In fact, since $\tau \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ maps exceptional chambers to exceptional chambers, it suffices to show that $(\tau(x), D) \geq 0$.

Let us first note that if $D \in \Delta_{\mathcal{P}}(X)$ satisfies $(D, i^*(D)) \geq 2$, then $(x, D) \geq 0$ for every $x \in \mathcal{C}_X^i$. Indeed, in this case we have

$$(D + i^*(D), D + i^*(D)) \geq 0.$$

Since $H^{1,1}(X,\mathbb{R})$ is hyperbolic, this implies that $(D+i^*(D))^\perp$ does not intersect the positive cone. On the other hand, we have $(h, D+i^*(D)) > 0$ for any invariant ample class h and therefore

$$0 \leq (x, D + i^*(D)) = 2(x, D),$$

where the last equality follows from $i^*(x) = x$.

We now consider the set

$$\Delta^i_{\mathcal{P}}(X) := \{D \in \Delta_{\mathcal{P}}(X) : (D, i^*(D)) \leq 1\}.$$

If $(D, i^*(D)) < 0$, then we have $D = i^*(D)$, since by [Bou04, Prop. 4.2] any prime divisors $E \neq E'$ satisfy $(E, E') \geq 0$. Hence for every $D \in \Delta^i_{\mathcal{P}}(X)$ we can define an isometry R_D by

$$R_D = \begin{cases} r_D & \text{if } D = i^*(D) \\ r_D \circ r_{i^*(D)} & \text{if } (D,\, i^*(D)) = 0 \\ r_{D+i^*(D)} & \text{if } (D,\, i^*(D)) = 1, \end{cases}$$

where for a class D with $(D,D) = -2$ the reflection r_D in the hyperplane orthogonal to D is given by

$$r_D(x) = x + (x, D)D.$$

By Theorem 1.4.4 we have $R_D \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ for every $D \in \Delta^i_{\mathcal{P}}(X)$. We want to show that $R_D \in \Gamma_i$ for every $D \in \Delta^i_{\mathcal{P}}(X)$ and furthermore that R_D acts on $H^2(X,\mathbb{R})^i$ by reflection in the hyperplane $D^\perp$.

In the cases $D = i^*(D)$ and $(D, i^*(D)) = 1$, the isometry R_D is given by reflection corresponding to an invariant class and we have

$$(D + i^*(D))^\perp = D^\perp \subset H^2(X,\mathbb{R})^i.$$

Now assume that $(D, i^*(D)) = 0$ and $x \in H^2(X,\mathbb{R})^i$. We have $(x, D) = (x, i^*(D))$ and hence

$$r_D \circ r_{i^*(D)}(x) = x + \frac{1}{2}(x, D + i^*(D))(D + i^*(D)).$$

This shows $r_D \circ r_{i^*(D)} \in \Gamma_i$ and, since $(D+i^*(D), D+i^*(D)) = -4$, that $r_D \circ r_{i^*(D)}$ acts on $H^2(X,\mathbb{R})^i$ as the reflection in the hyperplane

$$(D + i^*(D))^\perp = D^\perp \subset H^2(X,\mathbb{Z})^i.$$

Thus we have $\Gamma := \langle R_D : D \in \Delta^i_{\mathcal{P}}(X)\rangle \subset \Gamma_i$. Since Γ is a reflection group on the Lobachevsky space

$$\mathbb{H}^i := \{x \in \mathcal{C}^i_X : (x,x) = 1\},$$

the chamber

$$\mathcal{F} := \{x \in \mathbb{H}^i : (x, D) \geq 0 \text{ for all } D \in \Delta^i_{\mathcal{P}}(X)\}$$

is a fundamental domain for Γ by [VS93, Thm. 1.2]. Therefore, if x belongs to some invariant exceptional chamber, there exists a $\tau \in \Gamma$ such that $\tau(x) \in \mathcal{F}$, that is, $(\tau(x), D) \geq 0$ for all $D \in \Delta^i_{\mathcal{P}}(X)$. As shown above, we also have $(\tau(x), D) \geq 0$ for all $D \in \Delta_{\mathcal{P}}(X) \setminus \Delta^i_{\mathcal{P}}(X)$. Since τ maps exceptional chambers to exceptional chambers, we have $(\tau(x), D) > 0$ for all $D \in \Delta_{\mathcal{P}}(X)$ and hence $\tau(x) \in \mathcal{FE}^i_X$. □

Proposition 4.10.3. *Suppose that (X, i) and (Y, j) are two pairs of type M with $P_M(X, i) = P_M(Y, j)$. There exists a birational map $f : X \dashrightarrow Y$ with $j \circ f = f \circ i$.*

Proof. Let $\alpha : H^2(X, \mathbb{Z}) \to L$ and $\beta : H^2(Y, \mathbb{Z}) \to L$ be admissible markings such that $P_0(X, \alpha) = \sigma(P_0(Y, \beta))$ where $\sigma \in \Gamma(M)$. Then the map

$$g := \alpha^{-1} \circ \sigma \circ \beta : H^2(Y, \mathbb{Z}) \to H^2(X, \mathbb{Z})$$

is a Hodge isometry and a parallel transport operator. In particular we have $g(\mathcal{C}_Y) = \mathcal{C}_X$. Since g maps invariant exceptional chambers to invariant exceptional chambers, we can apply Lemma 4.10.2 to find an element $\tau \in \mathrm{Mon}^2_{\mathrm{Hdg}}(X)$ with $\tau \circ i^* = i^* \circ \tau$ such that $\widetilde{g} = \tau \circ g$ maps $\mathcal{FE}^j_Y$ to $\mathcal{FE}^i_X$ and hence also $\mathcal{FE}_Y$ to $\mathcal{FE}_X$. By Proposition 1.4.10, this shows that there exists a birational map $f : X \dashrightarrow Y$ with $f^* = \widetilde{g}$. Let

$$\widetilde{j} := f \circ i \circ f^{-1} : Y \dashrightarrow Y$$

be the induced birational involution. As in the proof of Theorem 4.9.5, one easily sees that $i^* \circ f^* = f^* \circ j^*$ and therefore $j^* = \widetilde{j}^*$. This shows that $\widetilde{j}^*$ fixes a Kähler class and thus $\widetilde{j}$ is biregular. By Theorem 4.1.3, we have $\widetilde{j} = j$. □

We now restrict to admissible sublattices of the form

$$M \subset L_{K3} \subset L = L_{K3} \oplus \langle -2 \rangle.$$

In this case M does not contain any classes with divisor 2, and therefore

$$\Delta(M) = \{\delta \in M : (\delta, \delta) = -2\}$$

We will see that there is only one deformation type of pairs of type M and that the period maps P_M and $P_{M,\mathcal{K}}$ are equivalent. Moreover, we show that in this case it is sufficient to remove hyperplanes orthogonal to -10-classes and use this to give an example where inseparable pairs do not exist. In that case, there exists a quasi-projective, and in particular Hausdorff moduli space for all pairs of type M (rather than only simple pairs).

Proposition 4.10.4. *Assume that M is an admissible sublattice of the form*

$$M \subset L_{K3} \subset L = L_{K3} \oplus \langle -2 \rangle.$$

Then there is only one deformation type of pairs of type M.

Proof. We need to show that Γ_M acts transitively on the Kähler-type chambers of M. For chambers that belong to the same connected component of $\widetilde{\mathcal{C}}_M$, this follows from [VS93, Thm. 1.2] as in the proof of Lemma 4.10.2. Here, it suffices to consider the reflection group

$$\Gamma := \langle r_\delta,\ \delta \in \Delta(M)\rangle \subset \Gamma_M.$$

Furthermore, by [Nik80b, Rem. 4.5.3], there exists an isometry $\xi : L \to L$ acting as the identity on M and with spinor norm -1 on $M^\perp$, that is $\xi(\Omega^+_{M^\perp}) = \Omega^-_{M^\perp}$. By Section 1.4.2, we have $\xi \notin O^+(L)$ and therefore

$$-\xi \in O^+(L) = \mathrm{Mon}(L),$$

which shows that $-\operatorname{id}_M \in \Gamma_M$ and hence the claim. □

Remark 4.10.5. The proof of Proposition 4.10.4 also shows that in this case the period map P_M is equivalent to $P_{M,\mathcal{K}}$ for any Kähler-type chamber $\mathcal{K}$: the composite map

$$\Omega^+_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}} \hookrightarrow \Omega_{M^\perp}/\Gamma_{M^\perp,\mathcal{K}} \to \Omega_{M^\perp}/\Gamma_{M^\perp}$$

is an isomorphism, which is compatible with P_M and $P_{M,\mathcal{K}}$. Indeed, assume that $\eta, \eta' \in \Omega^+_{M^\perp}$ are period points with $\eta = \sigma(\eta')$ for some $\sigma \in \Gamma(M) \subset \mathrm{Mon}(L)$. Then $\mathcal{K}$ and $\sigma(\mathcal{K})$ belong to the same connected component of $\widetilde{\mathcal{C}}_M$ and hence there exists some $\rho \in \Gamma$ in the reflection group defined above such that $\rho(\sigma(\mathcal{K})) = \mathcal{K}$. Since $\rho \in \widetilde{O}(M)$, it extends to an isometry of L acting trivially on $M^\perp$. Therefore $\eta = \rho(\sigma(\eta'))$, where $\rho \circ \sigma \in \Gamma(\mathcal{K})$. This shows injectivity. Surjectivity follows immediately from the existence of the isometry $-\xi \in \Gamma(M)$ given above.

Let $\Delta'_M(L) := \{\delta \in \Delta_M(L) : (\delta,\delta) = -10 \text{ and } \operatorname{div}\delta = 2\}$ and

$$\mathcal{D}''_M \subset \mathcal{D}'_M \subset \Omega_{M^\perp}/\Gamma_{M^\perp}$$

be the corresponding divisor.

Proposition 4.10.6. *Assume that M is as above and (X,i) and (Y,j) are two pairs of type M with*

$$P_M(X,i) = P_M(Y,j) \notin \mathcal{D}''_M.$$

Then $(X,i) \cong (Y,j)$.

Proof. By Proposition 4.10.3, there exists a birational map $f : X \dashrightarrow Y$ satisfying $f \circ j = i \circ f$, and therefore $f^*(\mathcal{FE}^j_Y) = \mathcal{FE}^i_X$. Inside the invariant fundamental exceptional invariant chamber, the invariant Kähler cone of (X,i) is cut out by walls

$$D^\perp \subset \mathcal{FE}^i_X, \qquad (D,D) = -10 \text{ and } \operatorname{div}(D) = 2.$$

By assumption about M, there exist no such divisors in $\Delta^i(X)$. Moreover, since $P_M(X,i) \notin \mathcal{D}''_M$, there is no such divisor $D \in \Delta(X) \setminus \Delta^i(X)$ with $D^\perp \cap \mathcal{C}^i_X \neq \emptyset$ by Lemma 4.6.5. Therefore, we have $\mathcal{FE}^i_X = \mathcal{K}^i_X$. The same is true for Y, hence f maps some Kähler class to a Kähler class. □

Example 4.10.7. Assume that $M \subset L_{K3}$ is unimodular, that is, one of the lattices U, $U \oplus E_8$ or $U \oplus 2E_8$. In this case we have

$$L = M \oplus M^\perp = M \oplus K \oplus \mathbb{Z}e,$$

where K is the orthogonal complement of M in L_{K3}. Any element $\delta \in L$ with $\mathrm{div}(\delta) = 2$ can be written as $\delta = 2\delta_M + \delta_{M^\perp}$ with $\delta_M \in M$ and $\delta_{M^\perp} \in M^\perp$. If $\delta \in \Delta'_M(L)$, then by Lemma 4.6.5, we have

$$(\delta_M, \delta_M) < 0, \quad (\delta_{M^\perp}, \delta_{M^\perp}) < 0, \quad 4(\delta_M, \delta_M) + (\delta_{M^\perp}, \delta_{M^\perp}) = -10.$$

This implies $(\delta_{M^\perp}, \delta_{M^\perp}) = -2$. Since $\delta^\perp = \delta^\perp_{M^\perp} \subset \Omega_{M^\perp}$, we have $\mathcal{D}''_M \subset \mathcal{D}_M$. Therefore,

$$P_M : \mathcal{M}_M \to (\Omega_{M^\perp}/\Gamma_{M^\perp}) \setminus \mathcal{D}_M$$

is bijective. In particular, $(\Omega_{M^\perp}/\Gamma_{M^\perp}) \setminus \mathcal{D}_M$ is a quasi-projective moduli space for all pairs of type M.

Remark 4.10.8. In [Cam13], Camere studies moduli spaces of lattice polarized irreducible symplectic manifolds, which is related to this chapter. Let $j : M \hookrightarrow L$ be a primitive embedding of any hyperbolic lattice. An M-polarization of an irreducible symplectic manifold is a lattice embedding $i : M \to \mathrm{Pic}(X)$. An (M,j)-polarized manifold is a pair (X, φ), where X is an M-polarized and $\varphi : H^2(X,\mathbb{Z}) \to L$ is a marking with $\varphi \circ i = j$. If $\mathrm{Pic}(X) = i(M)$, then (X,φ) is said to be strictly (M,j)-polarized. Moreover, for some fixed Kähler-type chamber $K(M)$ of M, the pair (X,φ) is called ample (M,j)-polarized, if $i(K(M))$ contains a Kähler class.

If $j(M)$ is an admissible sublattice, then an ample (M,j)-polarized pair admits a non-symplectic involution of type M. However, since the embedding $i : M \to \mathrm{Pic}(X)$ is a part of the object, the relevant subgroup $\Gamma_{M,j} \subset O^+(M^\perp)$ in this case is the restriction of

$$\mathrm{Mon}(M,j) := \{\sigma \in \mathrm{Mon}(L) : \sigma(m) = m \text{ for every } m \in M\}.$$

Among other results, Camere then shows that the period map induces a bijection

$$\mathcal{M}^{sa}_{M,j}/\,\mathrm{Mon}(M,j) \to \Omega'_{M^\perp}/\Gamma_{M,j},$$

where $\Omega'_{M^\perp} \subset \Omega^+_{M^\perp}$ is a dense connected subset which is invariant under $\Gamma_{M,j}$, and $\mathcal{M}^{sa}_{M,j}$ is the set of strictly ample (M,j)-polarized pairs contained in a connected component of the moduli space of ample (M,j)-polarized pairs.

Chapter 5

Invariant lattices of non-symplectic involutions

Recall that an admissible sublattice of

$$L = L_n = 3U \oplus 2E_8 \oplus \langle 2 - 2n \rangle$$

is a hyperbolic sublattice of the form $M = L^{\iota_M}$ for some involution $\iota_M \in \mathrm{Mon}(L)$. As a consequence of Theorem 4.8.11, a sublattice is admissible if and only if it is isometric to the invariant sublattice of a non-symplectic involution of a $K3^{[n]}$-type manifold.

In the case $n = 2$, admissible sublattices have been classified in [BCS14]. In this chapter we will give a partial classification in the case $n > 2$. We write

$$2n - 2 = 2^l \cdot m,$$

where $l \geq 1$ and $m \geq 1$ is odd. The assumption $n > 2$ implies that $l > 1$ or $m > 1$. By Proposition 2.1.2, the decomposition

$$A_L \cong \frac{\mathbb{Z}}{(2n-2)\mathbb{Z}} \cong \frac{\mathbb{Z}}{m\mathbb{Z}} \oplus \frac{\mathbb{Z}}{2^l\mathbb{Z}}$$

is orthogonal.

The results of this chapter are summarized in the following Theorem.

Theorem 5.0.1. *Let $M \subset L_n$ be an admissible sublattice. Then for some number $a \geq 0$, one of the following statements holds.*

(i) $A_M \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/(2n-2)\mathbb{Z}, \quad A_{M^\perp} \cong (\mathbb{Z}/2\mathbb{Z})^a.$

If $r(M) > 2$, then M and $M^\perp$ are among the lattices S and K, respectively, given in Proposition 5.2.1 (i) and (iii).

(ii) $A_M \cong (\mathbb{Z}/2\mathbb{Z})^a, \quad A_{M^\perp} \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/(2n-2)\mathbb{Z}.$

If $r(M^\perp) > 2$, then M and $M^\perp$ are among the lattices K and S, respectively, given in Proposition 5.2.1 (ii) and (iii).

(iii) $l = 1$ *and* $A_M \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/m\mathbb{Z}, \quad A_{M^\perp} \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/2\mathbb{Z}$.

If $r(M) > 2$, *then* M *and* $M^\perp$ *are among the lattices* S *and* K, *respectively, given in Propositions 5.3.2, 5.4.2, and Tables 5.1, 5.2 and 5.3 (where* $s_{(+)} = 1$*).*

(iv) $l = 1$ *and* $A_M \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/2\mathbb{Z}, \quad A_{M^\perp} \cong (\mathbb{Z}/2\mathbb{Z})^a \oplus \mathbb{Z}/m\mathbb{Z}$.

If $r(M^\perp) > 2$, *then* M *and* $M^\perp$ *are among the lattices* K *and* S, *respectively, given in Propositions 5.3.2, 5.4.1 5.4.2, and Tables 5.1, 5.2, 5.3 (where* $s_{(+)} = 2$*).*

In Section 5.1 we will prove the claims about the discriminant groups (Proposition 5.1.1). The classification of the lattices in the cases (i) and (ii) will be given in Section 5.2. A large part of the lattices in cases (iii) and (iv) will be covered in Section 5.3. The remaining lattices are considered case by case in Section 5.4.

As stated in Theorem 5.0.1, we will see that exactly one of the lattices M and $M^\perp$ is 2-elementary. It will be conventient to always denote this lattice by K and the other one by S. We will see that the cases $M = S$, $M^\perp = K$ and $M = K$, $M^\perp = S$ correspond to the cases $\iota_M \in \widetilde{O}(L)$ and $-\iota_M \in \widetilde{O}(L)$, respectively.

Lemma 5.0.2. *The map*

$$\iota \mapsto \begin{cases} \iota & \textit{if} \quad \iota \in \widetilde{O}(L) \\ -\iota & \textit{if} \ -\iota \in \widetilde{O}(L) \end{cases}$$

defines a bijection between involutions $\iota \in \mathrm{Mon}(L)$ *with hyperbolic invariant lattice, and involutions* $\tilde{\iota} \in \widetilde{O}(L)$ *with hyperbolic invariant or coinvariant lattice.*

Proof. Assume that $\iota \in \mathrm{Mon}(L)$. By Lemma 1.4.7, we have $\iota \in \widetilde{O}(L)$ or $-\iota \in \widetilde{O}(L)$ and by the assumption $n > 2$ only one of them can be true. Hence the map is well-defined and injective. On the other hand, for $\tilde{\iota} \in \widetilde{O}(L)$ as above, let

$$\iota := \begin{cases} \tilde{\iota}, & \text{if the invariant lattice of } \tilde{\iota} \text{ is hyperbolic,} \\ -\tilde{\iota} & \text{otherwise.} \end{cases}$$

It remains to show that $\iota \in O^+(L)$. Since the invariant lattice $M := L^\iota$ is hyperbolic, there exists a positive definite 1-dimensional subspace $W_M \subset M_\mathbb{R}$ and a positive definite 2-dimensional subspace $W_{M^\perp} \subset M_\mathbb{R}^\perp$. Then ι acts on $W_M \oplus W_{M^\perp} \subset L_\mathbb{R}$ with eigenvalues $1, -1, -1$. The claim follows from Lemma 1.4.6. $\square$

We will therefore only consider involutions $\tilde{\iota} \in \widetilde{O}(L)$ and denote by $S = L^{\tilde{\iota}}$ the invariant and by $K = S^\perp$ the coinvariant lattice. The signatures of S and K are denoted by $(s_{(+)}, s_{(-)})$ and $(k_{(+)}, k_{(-)})$, respectively. We will have to consider both the case $s_{(+)} = 1,\ k_{(+)} = 2$ and the case $s_{(+)} = 2,\ k_{(+)} = 1$. The admissible lattice $M = L^\iota$ is given by S in the first case and by K in the second case.

5.1 Discriminant group

In this section, we determine the discriminant groups of S and K.

If $l = 1$, then the discriminant form of L is given by

$$A_L \cong A_{\langle -2m\rangle} \cong q_m(2) \oplus q(m),$$

where

$$q_m(2) = \begin{cases} q_-(2), & \text{if } m \equiv 1 \pmod 4 \\ q_+(2), & \text{if } m \equiv 3 \pmod 4 \end{cases}$$

and $q(m) \cong \mathbb{Z}/m\mathbb{Z}$ is generated by an element of square $-\frac{2}{m}$. Together with $\operatorname{sign}(A_L) = -1 + 8\mathbb{Z}$, this implies

$$\operatorname{sign} q(m) = \begin{cases} 8\mathbb{Z} & \text{if } m \equiv 1 \pmod 4 \\ -2 + 8\mathbb{Z} & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Let $H_S \subset A_S$ and $H_K \subset A_K$ be the subgroups and $\gamma : H_S \to H_K$ be the isomorphism described in Section 2.2.

Proposition 5.1.1. *One of the following cases holds:*

(i) $A_S = H_S \oplus A_L, \quad A_K = H_K$, *or*

(ii) $l = 1$ *and* $A_S = H_S \oplus q(m), \quad A_K = H_K \oplus q_m(2)$.

Furthermore, $H_S \cong H_K(-1)$ *is 2-elementary.*

Proof. The isomorphism γ conjugates $\tilde{\imath}|_{H_S} = \mathrm{id}_{H_S}$ to $\tilde{\imath}|_{H_K} = -\mathrm{id}_{H_K}$ by Proposition 2.2.1. This shows that

$$H_L \cong H_S \cong H_K \cong \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^a, \qquad a \geq 0$$

as groups. Consider the isometry

$$\frac{H_L^\perp}{H_L} \cong A_L \cong \frac{\mathbb{Z}}{2^l\mathbb{Z}} \oplus \frac{\mathbb{Z}}{m\mathbb{Z}}. \tag{5.1}$$

Since m is odd and $\tilde{\imath}$ acts trivially on A_L, and since H_L is contained in the Sylow 2-subgroup of $A_S \oplus A_K$, this implies

$$A_S = (A_S)_2 \oplus \frac{\mathbb{Z}}{m\mathbb{Z}},$$

where $(A_S)_2 \subset A_S$ is the Sylow 2-subgroup.

Equation (2.1) gives

$$m \cdot 2^l = |A_L| = [A_S : H_S][A_K : H_K] = m[(A_S)_2 : H_S][A_K : H_K]. \tag{5.2}$$

First assume that $l = 1$. From equation (5.2), we obtain $(A_S)_2 = H_S$ or $A_K = H_K$. In either case, one the forms H_S or H_K is non-degenerate, and since $H_S \cong H_K(-1)$ also the other one. By Proposition 2.1.2, we have orthogonal decompositions

$$A_S = H_S \oplus H_S^\perp, \qquad A_K = H_K \oplus H_K^\perp.$$

The two cases correspond to (ii) and (i).

Now assume $l > 1$. Let $G := (A_S)_2 \oplus A_K$ and $G_2 \subset G$ be the 2-torsion subgroup. Since G contains an element of order at least 2^l, we have $[G : G_2] \geq 2^{l-1}$. On the other hand, since $H_S \oplus H_K \subset G_2$, equation (5.2) implies $[G : G_2] \leq 2^l$. We first show that $[G : G_2] = 2^l$ is impossible. Indeed, this would imply $H_S \oplus H_K = G_2$ and

$$G = \frac{\mathbb{Z}}{2^l\mathbb{Z}} \oplus \frac{\mathbb{Z}}{4\mathbb{Z}} \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2a-2} \quad \text{or} \quad G = \frac{\mathbb{Z}}{2^{l+1}\mathbb{Z}} \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{2a-1}.$$

In the first case, H_S or H_K would contain the 2-torsion element of $\mathbb{Z}/2^l\mathbb{Z}$, which contradicts (5.1).

In the second case, we can assume that

$$(A_S)_2 \cong \frac{\mathbb{Z}}{2^{l+1}\mathbb{Z}} \oplus \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{a-1} \qquad A_K = H_K.$$

As before, this would imply $A_S = H_S \oplus H_S^\perp$, a contradiction.

Consequently, we have $[G : G_2] = 2^{l-1}$ and therefore $G = \mathbb{Z}/2^l\mathbb{Z} \oplus H_S \oplus H_K$. Since $\tilde{\iota}$ acts trivially on A_L and $l > 1$, this shows $A_S = H_S \oplus A_L$ and $A_K = H_K$, which is case (i). □

Proposition 5.1.2. *Let $S \subset L$ be a primitive sublattice and $K := S^\perp$. If the discriminant groups of S and K are as in Proposition 5.1.1, then the involution*

$$\begin{aligned} S \oplus K &\to S \oplus K \\ (v, w) &\mapsto (v, -w) \end{aligned}$$

extends to an involution $\tilde{\iota} \in \widetilde{O}(L)$.

Proof. This follows immediately from Proposition 2.2.1, since $\mathrm{id}_{A_K} = -\mathrm{id}_{A_K}$. □

Assumption 1. For the rest of this chapter, we assume that $r(S) > 2$.

For $r(S) \leq 2$, the classification depends on arithmetic properties of m. For example, the lattice $M = \langle 2m \rangle$ is an admissible sublattice if and only if -1 is a quadratic residue modulo m.

Proposition 5.1.3. *The lattices S and K are unique in their genus and the homomorphisms $O(S) \to O(A_S)$ and $O(K) \to O(A_K)$ are surjective.*

Proof. Since K is 2-elementary and either indefinite or isometric to $\langle 2\rangle$ or $\langle 2\rangle \oplus \langle 2\rangle$, the claim is true by [Nik80b, Thm. 3.6.2 and Thm. 3.6.3].

Assume that $r(S) = l((A_S)_2)$ and that $(A_S)_2$ is not of the form

$$u(2) \oplus A' \quad \text{or} \quad v(2) \oplus A'$$

for some 2-elementary quadratic form A'. By Theorem 2.3.1 this would imply $l((A_S)_2) \leq 2$ and hence $r(S) \leq 2$, in contradiction to Assumption 1. In every other case the claim follows from Theorem 2.3.5. □

As discussed in Remark 4.2.3, admissible sublattices should be classified up to $\mathrm{Mon}(L)$ rather than $O(L)$. In fact, it follows from Assumption 1 that this is equivalent:

Proposition 5.1.4. *The homomorphism*

$$\{\sigma \in O(L) : \sigma(S) = S\} \to O(A_L)$$

is surjective.

Proof. We show the claim for case (i) of Proposition 5.1.1. For any $\alpha \in O(A_L)$ the isometry

$$\mathrm{id}_{H_S} \oplus \alpha \in O(A_S)$$

is induced by an isometry $\sigma \in O(S)$, by Proposition 5.1.3. It follows from Proposition 2.2.1, that the isometry $\sigma \oplus \mathrm{id}_K \in O(S \oplus K)$ extends to L. In case (ii), the proof is the same, since $O(A_L) = O(q(m))$. □

Therefore, the $O(L)$-orbits of S (and hence those of K) and its $\widetilde{O}(L)$-orbits coincide. By Lemma 1.4.7, the $\widetilde{O}(L)$-orbits and the $\mathrm{Mon}(L)$-orbits of S concide.

In the following sections we classify the sublattices $S \subset L$ and $K = S^\perp$ with discriminant as given in Proposition 5.1.1. In every case, we will split off a lattice with discriminant $q(m) \oplus q'$ and reduce the problem to 2-elementary sublattices of the unimodular lattices L_{K3} or $2U \oplus 2E_8$. A classification of 2-elementary lattices was given by Nikulin [Nik83, Thm. 4.3.2]. We will use the following 2-elementary lattices:

L	U	$U(2)$	A_1	D_{8k}	D_{8k+4}	E_7	E_8	$E_8(2)$
A_L	0	$u(2)$	$q_-(2)$	$u(2)$	$v(2)$	$q_+(2)$	0	$u(2)^4$

Lattices with invariants given below exist by Theorem 2.3.4. A list of their Gram matrices or Dynkin diagrams will be given at the end of this chapter.

N	$\operatorname{sign} N$	A_N	m
$U^{(m)}$	$(1,1)$	$q(m)$	$1+4\mathbb{Z}$
$V^{(m)}(2)$	$(1,1)$	$u(2) \oplus q(m)$	$1+8\mathbb{Z}$
$A_2^{(m)}$	$(0,2)$	$q(m)$	$3+4\mathbb{Z}$
$A_4^{(m)}(2)$	$(0,4)$	$u(2) \oplus v(2) \oplus q(m)$	$5+8\mathbb{Z}$
$D_4^{(m)}$	$(0,4)$	$v(2) \oplus q(m)$	$1+4\mathbb{Z}$
$E_6^{(m)}(2)$	$(0,6)$	$u(2)^2 \oplus v(2) \oplus q(m)$	$3+8\mathbb{Z}$
$B_2^{(m)}(2)$	$(0,2)$	$u(2) \oplus q(m)$	$7+8\mathbb{Z}$
$C_3^{(m)}$	$(2,1)$	$v(2) \oplus q_-(2) \oplus q(m)$	$3+4\mathbb{Z}$

For 2-elementary lattices, the existence Theorem 2.3.4 can be reformulated in the following way.

Lemma 5.1.5. *A lattice of signature $(l_{(+)}, l_{(-)})$ with 2-elementary discriminant form A exists if and only if*

(i) $l_{(+)}, l_{(-)} \geq 0$ *and* $l(A) \leq l_{(+)} + l_{(-)}$,

(ii) $l_{(+)} - l_{(-)} \equiv \operatorname{sign}(A) \pmod 8$,

(iii) $A \neq u(2)^b \oplus v(2)$ *if* $l(A) = l_{(+)} + l_{(-)} = 2b+2$.

Proof. Assume that $l(A) = l_{(+)} + l_{(-)}$ and $A \neq q_{\pm}(2) \oplus A'$. Then by Theorem 2.3.1, the form A can be written as $A \cong u(2)^b$ or $A \cong u(2)^b \oplus v(2)$ for some number $b \geq 0$. Since

$$\operatorname{discr} K(u(2)) \equiv 2^2 \pmod{(\mathbb{Z}_2^*)^2}$$

and

$$\operatorname{discr} K(v(2)) \equiv 3 \cdot 2^2 \not\equiv \pm 2^2 \pmod{(\mathbb{Z}_2^*)^2},$$

this shows the claim. □

We will frequently make use of the following fact.

Lemma 5.1.6. $l(H_S) \leq 11$.

Proof. This follows from $l(H_S) \leq r(S)$, $l(H_S) = l(H_K) \leq r(K)$ together with $r(S) + r(K) = 23$. □

5.2 Non-split discriminant

Proposition 5.2.1. *In case (i) of Proposition 5.1.1 we have either*

(i) $l = 1, \quad S = \langle 2\rangle \oplus 5\langle -2\rangle \oplus \langle -2m\rangle, \quad K = 2U \oplus 3D_4$, *or*

(ii) $l = 1, \quad S = 2\langle 2\rangle \oplus 6\langle -2\rangle \oplus \langle -2m\rangle, \quad K = U(2) \oplus 3D_4$, *or*

(iii) there exists a 2-elementary sublattice $S_0 \subset L_{K3}$ *such that* $S = S_0 \oplus \langle 2-2n\rangle$ *and* $K = S_0^\perp \subset L_{K3}$, *up to isometry of* L.

Proof. Assume that there exists a lattice S_0 of signature $(s_{(+)}, s_{(-)} - 1)$ with discriminant form H_S. Let K' be a lattice isometric to K. Since

$$\mathrm{sign}(S_0 \oplus K') = (3, 19)$$

and

$$A_{S_0} \cong H_S \cong H_K(-1) = A_K(-1) \cong A_{K'}(-1),$$

the unimodular lattice L_{K3} is an overlattice of $S_0 \oplus K'$. This defines an embedding

$$S' := S_0 \oplus \langle 2-2n\rangle \subset L$$

with orthogonal complement K'. Let $\gamma : H_S \to H_K$ and $\gamma' : H_{S'} \to H_{K'}$ be the corresponding isomorphisms. By Proposition 5.1.3, there exist isometries

$$\varphi : S \to S', \qquad \psi : K \to K'$$

with $\bar{\psi} \circ \gamma = \gamma' \circ \bar{\varphi}$ and we can apply Propositon 2.2.1 to see that the sublattices $S \subset L$ and $S' \subset L$ are isometric.

Now assume that there exists no such lattice S_0. We claim that $l = 1$ and $H_S \cong u(2)^b \oplus v(2)$, where $r(S) = 2b + 3$. Indeed, Theorem 2.3.4 implies that

$$l(A_S) = l(H_S) + 1 = r(S)$$

and that H_S is not of the form $q_\pm(2) \oplus H'$ for some quadratic form H'. If $l > 1$, then the latter is also true for A_S by [Nik80b, Cor. 1.9.2]. But then we have

$$K((A_S)_2) = K(H_S) \oplus K((A_L)_2)$$

by Theorem 2.3.3 and furthermore

$$|A_S| \equiv \mathrm{discr}\, K((A_S)_2) \pmod{(\mathbb{Z}_2^*)^2},$$

and since A_L is the discriminant of a rank 1 lattice also

$$|A_L| \equiv \mathrm{discr}\, K((A_L)_2) \pmod{(\mathbb{Z}_2^*)^2}.$$

This would imply

$$|H_S| \equiv \operatorname{discr} K(H_S) \pmod{(\mathbb{Z}_2^*)^2},$$

which contradicts the non-existence of S_0 and hence shows $l = 1$. The second claim follows from Lemma 5.1.5.

Now from $H_S \cong u(2)^b \oplus v(2)$ and $r(S) = 2b + 3$ it follows that

$$4 \equiv \operatorname{sign} H_S \equiv \operatorname{sign} A_S + 1 \equiv 2s_{(+)} - r(S) + 1 \equiv 2s_{(+)} - 2b - 2 \pmod 8$$

and hence $b \equiv s_{(+)} + 1 \pmod 4$, which implies $b = s_{(+)} + 1$ by Lemma 5.1.6. By Proposition 5.1.3, we have case (i) if $s_{(+)} = 1$ and (ii) if $s_{(+)} = 2$. □

We remark that in cases (i) and (ii) the lattice K is not embeddable into L_{K3}.

5.3 Split discriminant

We now consider case (ii) of Proposition 5.1.1, that is, $l = 1$ and

$$A_S = H_S \oplus q(m), \quad A_K = H_K \oplus q_m(2).$$

Lemma 5.3.1. *We have*

$$U \oplus \langle -2m \rangle \cong \begin{cases} U^{(m)} \oplus \langle -2 \rangle & \text{if } m \equiv 1 \pmod 4 \\ A_2^{(m)} \oplus \langle 2 \rangle & \text{if } m \equiv 3 \pmod 4. \end{cases} \tag{5.3}$$

Proof. This follows from the fact that the lattice $U \oplus \langle -2m \rangle$ is unique in its genus by Theorem 2.3.5. □

Assumption 2. If $m \equiv 1 \pmod 4$, we assume that there exist lattices S_0 of genus $(s_{(+)} - 1, s_{(-)} - 1, H_S)$ and K_0 of genus $(k_{(+)}, k_{(-)} - 1, H_K)$. Let

$$S' := S_0 \oplus U^{(m)}, \quad K' := K_0 \oplus \langle -2 \rangle.$$

Similarly, if $m \equiv 3 \pmod 4$, we assume that there exist lattices S_0 of genus $(s_{(+)}, s_{(-)} - 2, H_S)$ and K_0 of genus $(k_{(+)} - 1, k_{(-)}, H_K)$. Let

$$S' := S_0 \oplus A_2^{(m)} \quad K' := K_0 \oplus \langle 2 \rangle.$$

Proposition 5.3.2. *If Assumption 2 holds, there exists an embedding $S_0 \subset 2U \oplus 2E_8$ with $K_0 \cong S_0^{\perp}$. Let $S', K' \subset L$ be the embeddings induced by the isometry (5.3). There exists an isometry $\sigma \in O(L)$ with $\sigma(S) = S'$, $\sigma(K) = K'$.*

Proof. Since $\operatorname{sign}(S_0 \oplus K_0) = (2, 18)$ and

$$A_{S_0} \cong H_S \cong H_K(-1) \cong A_{K_0}(-1),$$

the unimodular lattice $2U \oplus 2E_8$ is an overlattice of $S_0 \oplus K_0$. The induced orthogonal embeddings $S' \subset L$ and $K' \subset L$ satisfy $H_{S'} = A_{S_0}$ and $H_{K'} = A_{K_0}$. Let

$$\gamma : H_S \to H_K, \qquad \gamma' : H_{S'} \to H_{K'}$$

be the induced isomorphisms. By Proposition 5.1.3, there exist isometries

$$\varphi : S \to S', \qquad \psi : K \to K'$$

such that $\bar{\psi} \circ \gamma = \gamma' \circ \bar{\varphi}$. The statement follows from Proposition 2.2.1. □

5.4 Remaining cases

It remains to classify the sublattices with

$$A_S = H_S \oplus q(m), \quad A_K = H_K \oplus q_m(2),$$

where Assumption 2 is false.

Since $s_{(+)}, s_{(-)}, k_{(+)} \geq 1$, by Lemma 5.1.5 this happens exactly in these cases:

(i) $m \equiv 1 \pmod 4$ and $k_{(-)} = 0$,

(ii) $m \equiv 3 \pmod 4$ and $s_{(-)} = 1$,

(iii) $l(H_K) = r(K) - 1$ and $H_K \cong u(2)^b \oplus v(2)$,

(iv) $l(H_S) = r(S) - 2$ and $H_S \cong u(2)^b \oplus v(2)$,

(v) $l(H_S) = r(S) - 1$,

(vi) $l(H_S) = r(S)$.

Proposition 5.4.1. *The cases (i) and (v) are impossible. In case (ii), we have*

$$S \cong C_3^{(m)}, \quad K \cong \langle 2 \rangle \oplus D_{12} \oplus E_7.$$

Proof. If $m \equiv 1 \pmod 4$ and $k_{(-)} = 0$, then $K = \langle 2 \rangle$ or $K = \langle 2 \rangle \oplus \langle 2 \rangle$, contradicting $A_K = H_K \oplus q_-(2)$, hence (i) is impossible.

Proposition 2.3.2 implies that $\operatorname{sign}(H_S) \equiv l(H_S) \pmod 2$ for any 2-elementary quadratic form, and since $\operatorname{sign} q(m) \equiv 0 \pmod 2$, this shows

$$\operatorname{sign}(H_S) \equiv \operatorname{sign}(A_S) \equiv r(S) \pmod 2,$$

and hence that $l(H_S) = r(S) - 1$ is impossible.

Now assume that $m \equiv 3 \pmod 4$ and $s_{(-)} = 1$. By Assumption 1, we have $s_{(+)} = 2$, and

$$\operatorname{sign}(H_S) \equiv \operatorname{sign} A_S - \operatorname{sign} q(m) \equiv 3 \pmod 8,$$

and therefore

$$H_S \cong v(2) \oplus q_-(2), \qquad H_K \cong v(2) \oplus q_+(2).$$

These genera correspond to the lattices given in the statement. □

5.4.1 The case $l(H_S) = r(S) - 2$

We have $H_S \cong u(2)^b \oplus v(2)$, where $l(A_S) = l(H_S) = 2b + 2$. If $m \equiv 1 \pmod 4$, then

$$4 \equiv \operatorname{sign} H_S \equiv \operatorname{sign} A_S \equiv 2s_{(+)} - r(M) \equiv 2s_{(+)} - (4 + 2b) \pmod 8,$$

which implies $b \equiv s_{(+)} \pmod 4$ and hence $b = s_{(+)}$ by Lemma 5.1.6. Similarly, if $m \equiv 3 \pmod 4$, then

$$4 \equiv \operatorname{sign} H_S \equiv \operatorname{sign} A_S + 2 \equiv 2s_{(+)} - (4 + 2b) + 2 \pmod 8,$$

and therefore $b = s_{(+)} + 1$. Since $H_K \cong H_S(-1)$, this determines the genera and hence the isometry classes of S and K (see Table 5.1).

$s_{(+)}$	m mod 8	S	K
1	1	$U(2) \oplus D_4^{(m)}$	$U \oplus U(2) \oplus D_4 \oplus E_8 \oplus \langle -2 \rangle$
	5	$U \oplus A_4^{(m)}(2)$	
	3	$U \oplus E_6^{(m)}(2)$	$U \oplus 3D_4 \oplus \langle 2 \rangle$
	7	$U(2) \oplus D_4 \oplus B_2^{(m)}(2)$	
2	1	$2U(2) \oplus D_4^{(m)}$	$U \oplus 3D_4 \oplus \langle -2 \rangle$
	5	$U \oplus U(2) \oplus A_4^{(m)}(2)$	
	3	$U \oplus U(2) \oplus E_6^{(m)}(2)$	$U(2) \oplus 2D_4 \oplus 3\langle -2 \rangle$
	7	$2U(2) \oplus D_4 \oplus B_2^{(m)}(2)$	

Table 5.1: The case $l(H_S) = r(S) - 2$

5.4.2 The case $l(H_S) = r(S)$

Proposition 5.4.2. *If $l(H_S) = r(S)$ and $\delta(H_S) = 1$, then*

$$S \cong s_{(+)}\langle 2 \rangle \oplus (s_{(-)} - 1)\langle -2 \rangle \oplus \langle -2m \rangle,$$

where $2 \leq s_{(-)} \leq 10$ if $s_{(+)} = 1$, and $1 \leq s_{(-)} \leq 9$ if $s_{(+)} = 2$. Moreover, in this case the lattice K is the unique 2-elementary lattice of signature $(3 - s_{(+)}, 20 - s_{(-)})$ with $\delta(A_K) = 1$ and $l(A_K) = s_{(+)} + s_{(-)} + 1$.

Proof. The lattice S is isometric to the lattice given above, since $(A_S)_2 = H_S$ is the unique 2-elementary quadratic form with $l(H_S) = r(S)$, $\delta(H_S) = 1$ and $\operatorname{sign} H_S \equiv \operatorname{sign} A_S - \operatorname{sign} q(m) \pmod 8$ by Theorem 2.3.1. If $s_{(+)} = 1$, then $s_{(-)} \geq 2$ is a consequence of Assumption 1. The upper bounds of $s_{(-)}$ follow from Lemma 5.1.6. The statement about K follows from $H_K \cong H_S(-1)$ and Theorem 2.3.1. $\square$

Now assume that $\delta(H_S) = 0$. Theorem 2.3.4 implies

$$H_S \cong \begin{cases} u(2)^b & \text{if } m \equiv \pm 1 \pmod 8 \\ u(2)^b \oplus v(2) & \text{if } m \equiv \pm 3 \pmod 8. \end{cases}$$

In the same way as before, we see that (mod 4)

$$b \equiv \begin{cases} s_{(+)} & \text{if } m \equiv 1 \pmod 8 \\ s_{(+)} + 1 & \text{if } m \equiv 5, 7 \pmod 8 \\ s_{(+)} + 2 & \text{if } m \equiv 3 \pmod 8. \end{cases}$$

If $m \equiv 1 \pmod 8$ and $s_{(+)} = 1$, then by Lemma 5.1.6 we have $b = 1$ or $b = 5$. If $b = 1$, then $r(S) = 2$, in contradiction to Assumption 1. In every other case we have $b \in \{s_{(+)}, s_{(+)} + 1, s_{(+)} + 2\}$ by Lemma 5.1.6. We obtain the lattices given in Table 5.2.

$s_{(+)}$	m mod 8	S	K
1	1	$E_8(2) \oplus V^{(m)}(2)$	$U \oplus U(2) \oplus E_8(2) \oplus \langle -2 \rangle$
	5	$U(2) \oplus A_4^{(m)}(2)$	$2U \oplus 3D_4 \oplus \langle -2 \rangle$
	3	$U(2) \oplus E_6^{(m)}(2)$	$U(2) \oplus 3D_4 \oplus \langle 2 \rangle$
	7	$U(2) \oplus B_2^{(m)}(2)$	$U \oplus 2D_8 \oplus \langle 2 \rangle$
2	1	$U(2) \oplus V^{(m)}(2)$	$U \oplus 2D_8 \oplus \langle -2 \rangle$
	5	$2U(2) \oplus A_4^{(m)}(2)$	$U(2) \oplus 3D_4 \oplus \langle -2 \rangle$
	3	$2U(2) \oplus E_6^{(m)}(2)$	$\langle 2 \rangle \oplus D_4 \oplus E_8(2)$
	7	$2U(2) \oplus B_2^{(m)}(2)$	$\langle 2 \rangle \oplus 2D_4 \oplus D_8$

Table 5.2: The case $l(H_S) = r(S)$

5.4.3 The case $l(H_K) = r(K) - 1$

In this case $H_K \cong u(2)^b \oplus v(2)$, where

$$r(K) = l(A_K) = l(H_K) + 1 = 2b + 3.$$

If $m \equiv 1 \pmod 4$, then

$$4 \equiv \operatorname{sign} H_K \equiv \operatorname{sign} A_K + 1 \equiv 2k_{(+)} - r(K) + 1 \equiv 2k_{(+)} - 2b - 2 \pmod 8,$$

and therefore $b \equiv k_{(+)} + 1 \pmod 4$, which by Lemma 5.1.6 implies $b = k_{(+)} + 1$. In the case $m \equiv 3 \pmod 4$, we have

$$4 \equiv \operatorname{sign} H_K \equiv \operatorname{sign} A_K - 1 \equiv 2k_{(+)} - 2b - 4 \pmod 8,$$

and hence $b = k_{(+)}$. The lattices are given in Table 5.3.

$s_{(+)}$	m mod 4	S	K
1	1	$U(2) \oplus 2D_4 \oplus D_4^{(m)}$	$2\langle 2\rangle \oplus 7\langle -2\rangle$
	3	$U(2) \oplus D_4 \oplus D_8 \oplus A_2^{(m)}$	$2\langle 2\rangle \oplus 5\langle -2\rangle$
2	1	$2U \oplus 2D_4 \oplus D_4^{(m)}$	$\langle 2\rangle \oplus 6\langle -2\rangle$
	3	$2U \oplus D_4 \oplus D_8 \oplus A_2^{(m)}$	$\langle 2\rangle \oplus 4\langle -2\rangle$

Table 5.3: The case $l(H_K) = r(K) - 1$

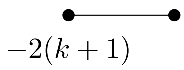
$U^{(m)}, \quad m = 4k+1$ $\qquad \begin{pmatrix} 2 & 1 \\ 1 & -2k \end{pmatrix}$

$V^{(m)}, \quad m = 8k+1$ $\qquad \begin{pmatrix} 4 & 1 \\ 1 & -2k \end{pmatrix}$

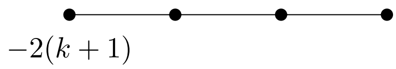
$B_2^{(m)}, \quad m = 8k+7$ $\qquad \begin{pmatrix} -4 & 1 \\ 1 & -2(k+1) \end{pmatrix}$

$C_3^{(m)}, \quad m = 2k+1$ $\qquad \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & -4k \end{pmatrix}$

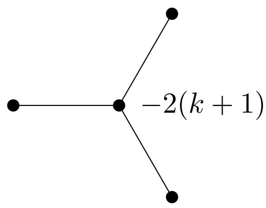
$A_2^{(m)}, \quad m = 4k+3$



$A_4^{(m)}, \quad m = 8k+5$



$D_4^{(m)}, \quad m = 4k+1$



$E_6^{(m)}, \quad m = 8k+3$

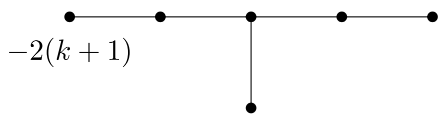


Table 5.4: Lattices with discriminant $\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^a \oplus \frac{\mathbb{Z}}{m\mathbb{Z}}$

Bibliography

[AS68] M.F. Atiyah and I.M. Singer. The index of elliptic operators: III. *Ann. Math.*, 87(3):546–604, 1968.

[AS08] M. Artebani and A. Sarti. Non-symplectic automorphisms of order 3 on K3 surfaces. *Math. Ann.*, 342(4):903–927, 2008.

[Ati67] M.F. Atiyah. *K-Theory.* W.A. Benjamin, Inc., New York, 1967.

[AV14a] E. Amerik and M. Verbitsky. Morrison-Kawamata cone conjecture for hyperkähler manifolds. 2014. arXiv:1408.3892.

[AV14b] E. Amerik and M. Verbitsky. Rational curves on hyperkähler manifolds. 2014. arXiv:1401.0479.

[BB66] W.L. Baily and A. Borel. Compactification of arithmetic quotients of bounded symmetric domains. *Ann. Math.*, 84(2):442–528, 1966.

[BCS14] S. Boissière, C. Camere, and A. Sarti. Classification of automorphisms on a deformation family of hyperkähler fourfolds by p-elementary lattices. 2014. arXiv:1402.5154.

[BD85] A. Beauville and R. Donagi. La varieté des droites d'une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985.

[Bea83a] A. Beauville. Some remarks on Kähler manifolds with $c_1 = 0$. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 1–26. Birkhäuser Boston, Boston, MA, 1983.

[Bea83b] A. Beauville. Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.*, 18(4):755–782, 1983.

[Bea11] A. Beauville. Antisymplectic involutions of holomorphic symplectic manifolds. *J. Topol.*, 4(2):300–304, 2011.

[BHT13] A. Bayer, B. Hassett, and Y. Tschinkel. Mori cones of holomorphic symplectic varieties of K3 type. 2013. arXiv:1307.2291.

[BM13] A. Bayer and E. Macrì. MMP for moduli of sheaves on K3s via wall-crossing: nef and movable cones, Lagrangian fibrations. 2013. arXiv:1301.6968.

[BNWS11] S. Boissière, M. Nieper-Wißkirchen, and A. Sarti. Higher dimensional Enriques varieties and automorphisms of generalized Kummer varieties. *J. Math. Pures Appl.*, 95(5):553–563, 2011.

[BNWS13] S. Boissière, M. Nieper-Wißkirchen, and A. Sarti. Smith theory and irreducible holomorphic symplectic manifolds. *J. Topol.*, 6(2):361–390, 2013.

[Bog78] F. Bogomolov. Hamiltonian Kähler manifolds. *Soviet Math. Dokl.*, 19:1462–1465, 1978.

[Bou04] S. Boucksom. Divisorial Zariski decompositions on compact complex manifolds. *Ann. Sci. École Norm. Sup. (4)*, 37(1):45–76, 2004.

[BS12] S. Boissière and A. Sarti. A note on automorphisms and birational transformations of holomorphic symplectic manifolds. *Proc. Amer. Math. Soc.*, 140(12):4053–4062, 2012.

[Cam12] C. Camere. Symplectic involutions of holomorphic symplectic fourfolds. *Bull. Lond. Math. Soc.*, 44(4):687–702, 2012.

[Cam13] C. Camere. Lattice polarized irreducible holomorphic symplectic manifolds. 2013. arXiv:1312.3224.

[Car35] H. Cartan. *Sur les groupes de transformations analytiques.* Hermann & cie, Paris, 1935.

[Cas78] J. W. S. Cassels. *Rational quadratic forms.* Academic Press London; New York, 1978.

[Deb84] O. Debarre. Un contre-exemple au théorème de torelli pour la variétés symplectiques irréductibles. *C. R. Acad. Sci. Paris*, 299:681–684, 1984.

[Fu13] L. Fu. Classification of polarized symplectic automorphisms of Fano varieties of cubic fourfolds. 2013. arXiv:1303.2241.

[Fuj87] A. Fujiki. On the de Rham cohomology group of a compact Kähler symplectic manifold. In *Algebraic Geometry, Sendai, 1985*, volume 10 of *Adv. Stud. Pure Math.*, pages 105–165. North-Holland, Amsterdam, 1987.

[GAL11] V. González-Aguilera and A. Liendo. Automorphisms of prime order of smooth cubic n-folds. *Arch. Math. (Basel)*, 97(1):25–37, 2011.

[GHS09] V. Gritsenko, K. Hulek, and G.K. Sankaran. Abelianisation of orthogonal groups and the fundamental group of modular varieties. *J. of Algebra*, 322:463–478, 2009.

[GHS13] V. Gritsenko, K. Hulek, and G.K. Sankaran. Moduli of K3 surfaces and irreducible symplectic manifolds. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 459–526. Int. Press, Somerville, MA, 2013.

[HT09] B. Hassett and Y. Tschinkel. Moving and ample cones of holomorphic symplectic fourfolds. *Geom. Funct. Anal.*, 19(4):1065–1080, 2009.

[Huy99] D. Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.

[Huy03] D. Huybrechts. The Kähler cone of a compact hyperkähler manifold. *Math. Ann.*, 326(3):499–513, 2003.

[Huy12] D. Huybrechts. A global Torelli theorem for hyperkähler manifolds [after M. Verbitsky]. *Astérisque*, (348):375–403, 2012. Séminaire Bourbaki: Vol. 2010/2011. Exposés 1027–1042.

[KV98] D. Kaledin and M. Verbitsky. Partial resolutions of Hilbert type, Dynkin diagrams, and generalized Kummer varieties. 1998. arXiv:9812078.

[Mar10] E. Markman. Integral constraints on the monodromy group of the hyperkähler resolution of the symmetric product of a K3 surface. *Internat. J. Math.*, 21(2):169–223, 2010.

[Mar11] E. Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322. Springer, Heidelberg, 2011.

[Mar13] E. Markman. Prime exceptional divisors on holomorphic symplectic varieties and monodromy reflections. *Kyoto J. Math.*, 53(2):345–403, 2013.

[Mon12] G. Mongardi. Symplectic involutions on deformations of $K3^{[n]}$. *Cent. Eur. J. Math.*, 10(4):1472–1485, 2012.

[Mon13] G. Mongardi. A note on the Kähler and Mori cones of manifolds of $K3^{[n]}$-type. 2013. arXiv:1307.0393.

[Mon14] G. Mongardi. Towards a classification of symplectic automorphisms on manifolds of $K3^{[n]}$ type. 2014. arXiv:1405.3232.

[MW14] G. Mongardi and M. Wandel. Induced automorphisms on irreducible symplectic manifolds. 2014. arXiv:1405.5706.

[Nam02] Y. Namikawa. Counter-example to global torelli problem for irreducible symplectic manifolds. *Math. Ann.*, 324(4):841–845, 2002.

[Nik80a] V.V. Nikulin. Finite automorphism groups of Kähler K3 surfaces. *Trans. Moscow Math. Soc.*, 38:71–135, 1980.

[Nik80b] V.V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Mathematics of the USSR-Izvestiya*, 14(1):103, 1980.

[Nik83] V.V. Nikulin. Factor groups of groups of automorphisms of hyperbolic forms with respect to subgroups generated by 2-reflections. *J. Soviet Math.*, 22(4):1401–1475, 1983.

[O'G97] K. O'Grady. The weight-two hodge structure of moduli spaces of sheaves on a K3 surface. *J. Alg. Geom.*, 6(4):599–644, 1997.

[O'G99] K. O'Grady. Desingularized moduli spaces of sheaves on a K3. *J. Reine Angew. Math.*, 512:49–117, 1999.

[O'G03] K. O'Grady. A new six-dimensional irreducible symplectic variety. *J. Alg. Geom.*, 12(3):435–505, 2003.

[OS01] K. Oguiso and J. Sakurai. Calabi-Yau threefolds of quotient type. *Asian J. Math.*, 5(1):43–77, 2001.

[OS11] K. Oguiso and S. Schröer. Enriques manifolds. *J. Reine Angew. Math.*, 661:215–235, 2011.

[OW13] H. Ohashi and M. Wandel. Non-natural non-symplectic involutions on symplectic manifolds of $K3^{[2]}$-type. 2013. arXiv:1305.6353.

[Oxb00] W.M. Oxbury. Varieties of maximal line subbundles. *Math. Proc. Camb. Phil. Soc.*, 129(9):9–18, 2000.

[Pet86] C.A.M. Peters. Some applications of the Lefschetz fixed point theorems in (complex) algebraic geometry. *Cont. Math.*, 58(I), 1986.

[Sul77] D. Sullivan. Infinitesimal computations in topology. *Publications Mathématiques de l'IHÉS*, 47:269–331, 1977.

[Ver96] M. Verbitsky. Cohomology of compact hyper-Kähler manifolds and its applications. *Geom. Funct. Anal.*, 6(4):601–611, 1996.

[Ver13] M. Verbitsky. A global torelli theorem for hyperkähler manifolds. *Duke Math. J.*, 162(15):2929–2986, 2013.

[VS93] È.B. Vinberg and O.V. Shvartsman. Discrete groups of motions of spaces of constant curvature. In *Geometry, II*, volume 29 of *Encyclopaedia Math. Sci.*, pages 139–248. Springer, Berlin, 1993.

[Wol67] J.A. Wolf. *Spaces of Constant Curvature*. McGraw-Hill, Inc., New York, 1967.

[Yos04] K. Yoshikawa. $K3$ surfaces with involution, equivariant analytic torsion, and automorphic forms on the moduli space. *Invent. Math.*, 156(1):53–117, 2004.

Lebenslauf

Persönliche Daten

Name	Malek Joumaah
Geburtsdatum	09.08.1986
Geburtsort	Hameln

Ausbildung

24.06.2005	Abitur am Schiller-Gymnasium Hameln
10.2005–06.2010	Studium der Mathematik an der Leibniz Universität Hannover
11.10.2007	Vordiplom in Mathematik
01.06.2010	Diplom in Mathematik
	Thema der Diplomarbeit: Linearsysteme auf Enriquesflächen
seit 01.10.2010	Promotionsstudium an der Leibniz Universität Hannover

Tätigkeiten

10.2008–07.2010	Studentische Hilfskraft an der Leibniz Universität Hannover
seit 01.10.2010	Wissenschaftlicher Mitarbeiter am Institut für Algebraische Geometrie, Leibniz Universität Hannover